The Neoclassical Growth Model: Revisited using Recursive Methods

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We revisit the planner's problem in the Neoclassical Growth Model (NGM), a.k.a. the Ramsey-Cass-Koopmans model, in discrete time. The NGM forms the backbone for all modern macroeconomic business cycle models. We first consider the NGM in the form of the *sequence problem*, but then reconsider it through the lens of *dynamic programming*. This lecture follows what we have studied already in Math Methods.

1 The Neoclassical Growth Model (Ramsey-Cass-Koopmans)

Consider the planner's problem in the Neoclassical Growth Model in discrete time. We assume time is discrete and indexed by $t = 0, 1, \ldots, \infty$. We write the planner's problem as follows.

Planner's Problem. Given an initial level of capital, $k_0 > 0$, the social planner chooses an infinite sequence for consumption and capital,

$$\{c_t, k_{t+1}\}_{t=0}^{\infty},$$

so as to maximize the utility of the representative household:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t) \tag{1}$$

with $\beta \in (0, 1)$, subject to the resource constraint,

$$c_t + k_{t+1} = f(k_t) + (1 - \delta)k_t, \qquad \forall t \geq 0 \tag{2}$$

and non-negativity constraints,

$$c_t \geq 0, \qquad k_{t+1} \geq 0, \qquad \forall t \geq 0.$$

We call this problem the *sequence problem*. Note that I am already writing everything in intensive form: $k = K/L$, $y = Y/L$, $c = C/L$ and

$$f(k) \equiv F(k, 1) = F(K/L, 1)$$

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is the intensive form production function.

Note further that I am already writing the aggregate resource constraint with equality: if it were not satisfied at the planner's optimum with equality, then resources would have been left unused. The social planner could have raised social welfare simply by letting the household consume the unused resources.

Assumption 1. $\beta \in (0,1)$. By the Neoclassical assumptions on $F(K,L)$, the intensive form production function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, twice-differentiable, strictly increasing, and strictly concave:

$$f(0) = 0, \qquad f'(k) > 0, \qquad f''(k) < 0$$

and satisfies the Inada conditions:

$$\lim_{k \to 0} f'(k) = \infty, \qquad \text{and} \qquad \lim_{k \to \infty} f'(k) = 0.$$

For preferences, we make the standard regularity assumptions on $U : \mathbb{R}_+ \to \mathbb{R}$. That is, it is continuous, twice-differentiable, strictly increasing and strictly concave:

$$U'(c) > 0, \qquad U''(c) < 0$$

and satisfies the Inada conditions:

$$\lim_{c \to 0} U'(c) = \infty, \qquad \text{and} \qquad \lim_{c \to \infty} U'(c) = 0.$$

2 The finite-horizon solution using the Lagrangian Method

One way of solving the sequence problem is to use the Lagrangian method. The main technicality one must be aware of, however, is that there are issues when dealing with infinite spaces.

By this I mean that what one typically does is set up the Lagrangian as if the horizon were finite: $t \in \{0, 1, ...T\}$ for some finite but large $T \geq 1$. One can then take a "hand-waving" limit as T approaches infinity. However, one needs to formally prove that this is the unique and correct solution to the planner's problem.

Let me start with finite horizon version of the planner's problem.

Planner's Problem. Let $T < \infty$. The planner solves

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t U(c_t) \tag{3}$$

with $\beta \in (0,1)$, subject to the resource constraint,

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t, \qquad \forall t \in \{0, 1, ...T\}, \tag{4}$$

and non-negativity constraints,

$$c_t \geq 0, \qquad k_{t+1} \geq 0, \qquad \forall t \in \{0, 1, ...T\}. \tag{5}$$

Proposition 1. *The solution to the finite-horizon planner's problem exists and is unique. That is, there exists a unique path* $\{c_t, k_{t+1}\}_{t=0}^T$ *that maximizes the social welfare function* (3) *over the set of feasible allocations, i.e. allocations that satisfy* (4) *and* (5).

Proof. By continuity of U, the social welfare function (3) is continuous. Next, it is easy to verify that the set of feasible allocations $\{c_t, k_{t+1}\}_{t=0}^T$ is compact. By Weierstrass's theorem we guarantee the *existence* of a solution to the planner's problem.

Next by strict concavity of U, the social welfare function (3) is strictly concave. One can verify that the set of feasible allocations $\{c_t, k_{t+1}\}_{t=0}^T$ is convex due to the strict concavity of f. Therefore this is a convex optimization problem: we are maximizing a strictly concave function over a convex set. This implies that if a solution exists, it is *unique*.

Together, we conclude that the solution to the finite-horizon planner's problem exists and is unique. □

Let me now characterize the solution to the finite-horizon planner's problem. Letting $\beta^t \lambda_t$ denote the Lagrange multiplier on the period-t resource constraint (2), we have that the Lagrangian of the social planner's problem is given by:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t U(c_t) - \sum_{t=0}^{T} \beta^t \lambda_t \left[c_t + k_{t+1} - (1-\delta)k_t - f(k_t)\right]$$

We can then rewrite the Lagrangian as

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \left\{U(c_t) - \lambda_t \left[c_t + k_{t+1} - (1-\delta)k_t - f(k_t)\right]\right\}$$

Note that λ_t measures the planner's period-t shadow value of period-t resources—in short, it is the marginal social value of resources at time t.

We henceforth assume an interior solution. As long as $k_0 > 0$, an interior solution is indeed ensured by the Inada conditions on f and U.

First, for any $t \in \{0, ..., T\}$, the FOC with respect to c_t gives us

$$\frac{\partial \mathcal{L}}{\partial c_t} = U'(c_t) - \lambda_t = 0.$$

Second, for any $t \in \{0, ..., T-1\}$, the FOC with respect to k_{t+1} gives us

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = -\lambda_t + \beta \lambda_{t+1} \left[1 - \delta + f'(k_{t+1})\right] = 0.$$

Third, the FOC with respect to λ_t simply gives us back the resource constraint for period t:

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t.$$

As noted before, the Lagrange multiplier λ_t measures the marginal value of resources in period t: if we exogenously give the economy ϵ units of the good in period t, where ϵ is small enough, welfare, as evaluated from that period on, increases by approximately $\lambda_t\epsilon$. (Welfare as of period 0 increases by $\beta^t\lambda_t$, that is, by the discounted value of λ_t.) We thus have that

$$\lambda_t = U'(c_t).$$

Combining this with the FOC with respect to capital, we obtain the intertemporal Euler equation for the planner's problem:

$$\frac{U'(c_t)}{\beta U'(c_{t+1})} = 1 + f'(k_{t+1}) - \delta. \tag{6}$$

This condition has the following interpretation. The planner finds it optimal in every period to equate the marginal rate of substitution (MRS) between consumption today and consumption tomorrow with the corresponding marginal rate of transformation (MRT), which is simply the marginal product of capital net of depreciation (plus one).

Note that we have taken the FOCs with respect to k_{t+1} for all t up to $T-1$. What about $t = T$? That is, what about the optimal choice of k_{T+1}? The Karush-Kuhn-Tucker conditions with respect to k_{T+1} give us

$$\frac{\partial \mathcal{L}}{\partial k_{T+1}} \geq 0 \qquad \text{and} \qquad k_{T+1} \geq 0, \quad \text{with complementary slackness.}$$

Equivalently:

$$\beta^T \lambda_T \geq 0 \qquad \text{and} \qquad k_{T+1} \geq 0 \qquad \text{with} \quad \beta^T \lambda_T k_{T+1} = 0.$$

The latter means that either $k_{T+1} = 0$, or it better be that the shadow value of k_{T+1} is zero because there is no use of carrying capital into period $T+1$ (when the household is dead).

Given that $\lambda_T > 0$, that is, the marginal value of consumption is strictly positive in the terminal period T, this implies that the planner's optimum is the corner solution:

$$k_{T+1} = 0.$$

i.e. the non-negativity constraint $k_{T+1} \geq 0$ is binding. Intuitively, if k_{T+1} were strictly positive, the planner would be leaving unused resources on the table, which cannot be optimal (given strictly positive marginal utility of consumption).

Summing up, we reach the following characterization of the solution to the planner's problem.

Proposition 2. *The socially optimal allocation of consumption and capital,* $\{c_t, k_{t+1}\}_{t=0}^T$*, is the* *unique path that solves the following dynamic system:*

$$\frac{U'(c_t)}{\beta U'(c_{t+1})} = 1 + f'(k_{t+1}) - \delta, \tag{7}$$

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t, \tag{8}$$

along with the following boundary conditions:

$$k_0 > 0 \qquad \textit{given}, \qquad \textit{and} \qquad k_{T+1} = 0. \tag{9}$$

Note that this is a system of two first-order difference equations (7) and (8) in two variables, c_t and k_t. In general, such a system admits multiple solutions: there are multiple paths of c_t and k_t that satisfy these two difference equations. What is claimed above is that *only one* solution satisfies the two relevant boundary conditions, $k_0 > 0$ given and $k_{T+1} = 0$, and that this particular solution identifies the *unique* socially optimal plan.

Finally, note that the above two difference equations (7) and (8) have very simple interpretations. Condition (8) is the resource constraint, summarizing feasibility of the allocation (aside from the non-negativity constraints). Condition (7) is the Euler equation; it is derived from the FOCs of the planner's problem, and therefore summarizes *optimality*. Furthermore, the condition that $k_{T+1} = 0$ also an *optimality* condition. It says that the planner sees no value in holding capital past the terminal date T.

3 The infinite-horizon solution using the Lagrangian Method

For the infinite horizon solution to the planner's problem, one can set up the Lagrangian again but with $T \to \infty$. The Euler equation derived in 6 is the same. What changes, however, is the terminal condition, as there is now no more terminal period.

Recall the Karush-Kuhn-Tucker conditions with respect to k_{T+1}:

$$\beta^T \lambda_T \geq 0 \qquad \text{and} \qquad k_{T+1} \geq 0 \qquad \text{with} \qquad \beta^T \lambda_T k_{T+1} = 0.$$

Furthermore, recall that the Lagrange multiplier λ_T satisfies:

$$\lambda_T = U'(c_T)$$

that is, λ_T is equal to the marginal utility of consumption in the terminal period. The Kuhn-Tucker conditions therefore imply that either $k_{T+1} = 0$, or it better be that the shadow value of k_{T+1} is zero because there is no use of carrying capital into period $T+1$ (when the household is dead).

Let's now take our "hand-waving" limit of this condition as $T \to \infty$. In this case we replace of the terminal condition of the finite horizon problem (namely, $k_{T+1} = 0$), with what we call the *transversality* condition:

$$\lim_{T\to\infty} \beta^T \lambda_T k_{T+1} = 0,$$

which means that the (discounted) shadow value of capital converges to zero. Equivalently:

$$\lim_{T\to\infty} \beta^T U'(c_T) k_{T+1} = 0. \tag{10}$$

The transversality condition in (10) then becomes a necessary condition for pinning down the unique optimal allocation.

Proposition 3. *The socially optimal allocation of consumption and capital, $\{c_t, k_{t+1}\}_{t=0}^{\infty}$, is the unique path that solves the following dynamic system:*

$$\frac{U'(c_t)}{\beta U'(c_{t+1})} = 1 + f'(k_{t+1}) - \delta, \tag{11}$$

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t, \tag{12}$$

along with the following boundary conditions:

$$k_0 > 0 \qquad \textit{given}, \qquad \textit{and} \qquad \lim_{t\to\infty} \beta^t U'(c_t) k_{t+1} = 0. \tag{13}$$

Proposition 3 is similar to Proposition 2. It is the same system of two difference equations, (11) and (12), in two variables, c_t and k_t. In general, such a system admits multiple solutions: there are multiple paths of c_t and k_t that satisfy these two difference equations. The only difference between Proposition 3 and 2 is that we replace the terminal condition with the transversality condition (13).

It is important to note that the transversality condition, like the Euler equation, is also an *optimality* condition. It says that in the long run, it better be that either capital is converging to zero, or the discounted value of this capital converges to zero.

What is claimed above is that only one solution satisfies the two relevant boundary conditions—the initial condition for capital, k_0, and the transversality condition—and that this particular solution identifies the *unique* socially optimal plan.

Have we proven this solution exists and is unique as in the finite-horizon case? Not really. But let's prove it now using dynamic programming tools.

4 The Bellman Equation

We now write the planner's problem using the functional equation (see notes from Math Methods). Let

$$\gamma(k_t) \equiv f(k_t) + (1-\delta)k_t \tag{14}$$

denote the total goods available in period t. Then $c_t = \gamma(k_t) - k_{t+1}$.

We can thus write the planner's problem in the infinite-horizon NGM as

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(\gamma(k_t) - k_{t+1})$$

subject to

$$0 \leq k_{t+1} \leq \gamma(k_t)$$

with $k_0 > 0$ given.

Let $\mathbb{R}_+$ (the set of non-negative reals) be the set of all possible values for k. We can write the Bellman (functional) equation as:

$$v(k) = \max_{k' \in \Gamma(k)} \left[U(\gamma(k) - k') + \beta v(k') \right] \tag{15}$$

where the feasibility correspondence Γ is defined by:

$$\Gamma(k) = [0, \gamma(k)] \subseteq \mathbb{R}_+.$$

From the results presented in Math Methods on Dynamic Programming (Lecture 7), a solution v to the functional equation in (15) exists and is unique. Given v we can define the *policy function* g by

$$g(k) \equiv \{k' \in \Gamma(k) | v(k) = U(\gamma(k) - k') + \beta v(k')\}.$$

Given our policy function for optimal capital accumulation, $k' = g(k)$, we have that consumption is given by:

$$c = \gamma(k) - g(k) = f(k) + (1 - \delta)k - g(k).$$

From the results presented in Math Methods on Dynamic Programming (Lecture 7), we know the following:

Proposition 4. *A solution v to the functional equation in* (15) *exists and is unique. Furthermore:*
(i) v is continuous, strictly increasing, and strictly concave.
(iii) v is twice differentiable at all $k > 0$.
(iv) g is a single-valued, continuous function.

The twice-differentiability of v comes from the twice-differentiability of U and f. Of course, the one caveat is that we don't have a bounded return function (as we assumed in those notes). Recall that the per-period return function in this model is given by

$$\varphi(k, k') = U(\gamma(k) - k') = U(f(k) + (1 - \delta)k - k').$$

so we don't exactly satisfy the boundedness assumption. One way to take care of this is by restricting the state space for k to be a compact subset of $\mathbb{R}_+$. This is quite natural in most settings, and implies that φ is bounded on the compact set. Then we would be fine. If this is not the case, then please see details in Sections 4.3-4.4 of Stokey, Lucas and Prescott (1989).

In any case, let us continue with what we know from Proposition 4. Given that v is twice-differentiable, we can now use calculus in order to characterize g. If we take the first-order condition of the Bellman equation in (15), we have that

$$-U'(\gamma(k) - k') + \beta v'(k') = 0$$

which we may rewrite as

$$U'(\gamma(k) - k') = \beta v'(k').$$

The Benveniste-Scheinkman condition gives us

$$v'(k) = \gamma'(k)U'(\gamma(k) - k') = [f'(k) + (1 - \delta)]U'(f(k) + (1 - \delta)k - k') \tag{16}$$

The first of these conditions equates the marginal utility of consuming current output to the marginal value of allocating it to capital. The second condition states that the marginal value of current capital, in terms of total discounted utility, is given by the marginal *physical* return on capital carried into this period in terms of the physical good, times the marginal utility of consumption from that good.

Using these conditions we now state and prove the following result.

Proposition 5. *The policy function* $g(k)$ *is strictly increasing in* k.

Proof. To prove this, consider the first-order condition:

$$U'(f(k) + (1 - \delta)k - k') = \beta v'(k') \tag{17}$$

We know that $U, f,$ and v are all strictly increasing, strictly concave functions and twice-differential functions. Consider graphing both sides of equation (17) as a function of k'. Both sides are positively-valued. Due to concavity, the right-hand side corresponds to a downward-sloping function of k'. On the other hand, the left-hand side corresponds to an upward-sloping function of k'. Where the two curves intersect indicates the optimal choice of $k' = g(k)$. See Figure 1.

Now suppose we increase the state variable to $\hat{k} > k$. This only affects the left-hand side of equation (17). Clearly the term $f(k) + (1 - \delta)k$ would increase, and by concavity of U this would imply that the left-hand side of equation (17) would shift down, or rightwards.

Given this shift, the new intersection of these curves would result in an increase in the optimal level of capital:

$$g(\hat{k}) > g(k).$$

as was to be shown. □

Another way to prove that g is strictly increasing would be to take second derivatives and use the implicit function theorem. You can do this (although I find it less intuitive/instructive.)

Next, one may combine the FOC in (17) with the Benveniste-Scheinkman condition in (16) evaluated at k'. This gives us

$$U'(\gamma(k) - g(k)) = [f'(k') + (1 - \delta)]U'(\gamma(k') - g(k')),$$

or in other words

$$U'(c) = \beta[f'(k') + (1 - \delta)]U'(c').$$

That is, we get the intertemporal optimality condition! Also known as the Euler equation.

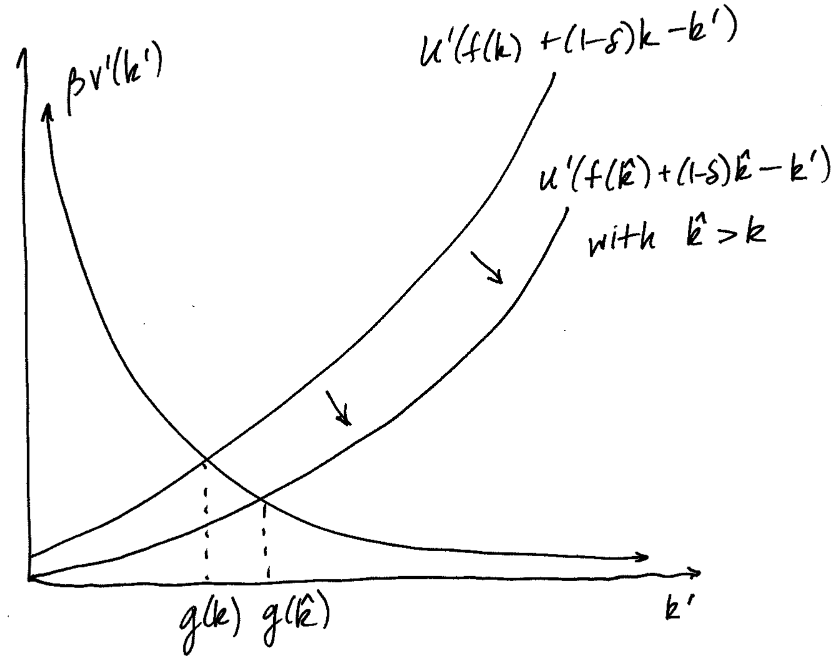

Figure 1. monotonicity of the policy function g

5 Steady State and Stability in the NGM

We now move on to study the dynamics of the optimal path, using what we have established for the policy function g. In particular, we first characterize the steady state(s) of the economy, and then study the transitional dynamics of the economy towards (or away) from its steady state(s).

Definition 1. A steady state is a point k^* such that $k^* = g(k^*)$.

By the assumption that capital is essential for production, there is always a trivial steady state at zero capital: $g(0) = f(0) = 0$. This steady state can be ruled out by assuming $k_0 > 0$; henceforth I will ignore the trivial zero capital case.

More interestingly, let us check whether there is a non-zero steady state with $k^* = g(k^*) > 0$.

Proposition 6. *There exists a unique non-zero steady state. At this steady state, the capital stock is given by $k = k^*$, where k^* is the unique solution to*

$$1 = \beta[1 - \delta + f'(k^*)] \tag{18}$$

or, equivalently,

$$f'(k^*) - \delta = \rho \equiv \frac{1}{\beta} - 1 > 0$$

where ρ is the discount rate; and steady-state consumption is given by

$$c^* = f(k^*) - \delta k^*. \tag{19}$$

Proof. In any such steady state, the following must hold:

$$U'(\gamma(k^*) - g(k^*)) = \beta[f'(k^*) + (1-\delta)]U'(\gamma(k^*) - g(k^*))$$

Using the fact that $U' > 0$, we infer that

$$\beta\left[f'(k^*) + (1-\delta)\right] = 1. \tag{20}$$

We can rewrite this as:

$$f'(k^*) = \frac{1}{\beta} - 1 + \delta$$

Recall that f' is continuous and strictly decreasing, and from the Inada conditions, $f'(k) \to \infty$ as $k \to 0$ and $f'(k) \to 0$ as $k \to \infty$. It follows that there exists a unique $k^* > 0$ that solves (20), proving both the existence and uniqueness of a non-zero steady state. See Figure 2.

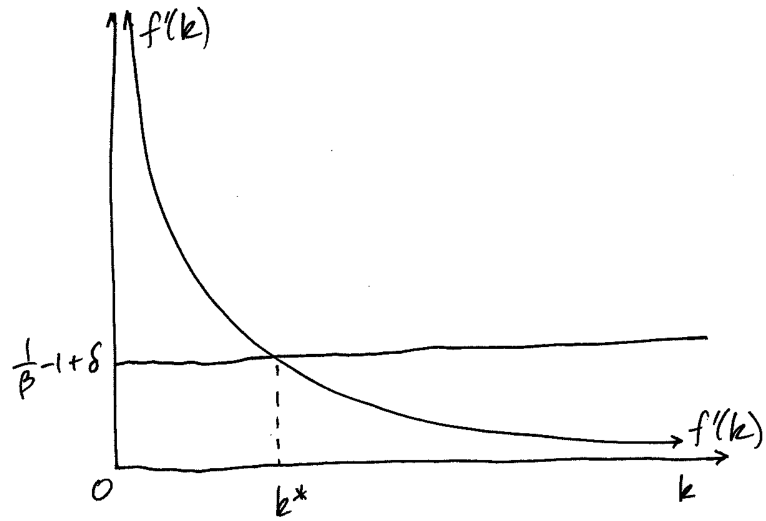

Figure 2. existence and uniqueness of steady state k^*

By the resource constraint, the steady-state level of consumption is given by

$$c^* = f(k^*) + (1-\delta)k^* - k^* = f(k^*) - \delta k^*$$

as in (19). □

The intuition for condition (18) is that in the steady state, the social return to saving (the marginal product of capital net of depreciation) is equated with the social cost of saving (the discount rate). Furthermore, in the steady state, gross investment is δk^*, that is, investment is just enough to offset the depreciation in capital so that capital remains constant.

We can furthermore do some trivial comparative statics on steady state capital.

Proposition 7. *The steady state level of capital k^* is decreasing in both ρ and δ.*

Proof. Steady state capital is the unique solution to

$$f'(k^*) = \rho + \delta.$$

As f' is continuous and decreasing, k^* is decreasing in ρ and δ. □

Next, let us study whether this steady state is locally or globally stable. Towards this goal, we state and prove the following lemma.

Lemma 1. *The unique steady state level of capital k^* that satisfies (20) and the optimal policy function g satisfy:*

$$k < g(k) \qquad \text{if and only if} \qquad k < k^*.$$

Proof. Strict concavity of v implies

$$v'(k) > v'(g(k)) \qquad \text{if and only if} \qquad k < g(k)$$

But if $k > 0$, we also have

$$\begin{aligned} v'(k) &= [f'(k) + (1-\delta)]U'(f(k) + (1-\delta)k - g(k)) \\ \beta v'(g(k)) &= U'(f(k) + (1-\delta)k - g(k)) \end{aligned}$$

by, respectively, the Benveniste-Scheinkman condition and the FOC. Combining these with the above if and only if statement, we infer that

$$\beta[f'(k) + (1-\delta)]U'(f(k) + (1-\delta)k - g(k)) > U'(f(k) + (1-\delta)k - g(k)) \qquad \text{iff} \qquad k < g(k).$$

Therefore,

$$\beta[f'(k) + (1-\delta)] > 1 \qquad \text{iff} \qquad k < g(k).$$

Using the fact that f' is strictly decreasing and that $\beta[f'(k^*) + (1-\delta)] = 1$, we conclude that

$$k < k^* \qquad \text{iff} \qquad k < g(k).$$

as was to be shown. □

Now, suppose that $k_0 \in (0, k^*)$ and construct the optimal sequence by letting $k_{t+1} = g(k_t)$ for all t. Note that this sequence is increasing and bounded from above by k^*. It follows that the sequence converges. By continuity of g, the limit is $g(k^*)$. A similar argument applies if we start with $k_0 > k^*$. We thus reach the following conclusion.

Proposition 8. *The steady state is globally stable and the transition is monotonic in the sense that capital increases over time if $k_0 < k^*$, and decreases over time if $k_0 > k^*$.*

We can thus restate the solution to the planner's problem as follows:

Proposition 9. *The socially optimal allocation of consumption and capital,* $\{c_t, k_{t+1}\}_{t=0}^{\infty}$*, is the unique path that solves the following dynamic system:*

$$\frac{U'(c_t)}{\beta U'(c_{t+1})} = 1 - \delta + f'(k_{t+1}), \tag{21}$$

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t, \tag{22}$$

with the following boundary conditions:

$$k_0 > 0 \qquad \textit{given}, \qquad \textit{and} \qquad \lim_{t\to\infty} k_t = k^*. \tag{23}$$

Note that the transversality condition has been replaced with the requirement that $k_t \to k^*$. In fact, it is easy verify that the transversality condition holds since $k_t \to k^*$ and $c_t \to c^*$.

References

Stokey, Nancy L., Robert E. Lucas, and Edward C. Prescott, *Recursive Methods in Economic Dynamics*, Harvard University Press, 1989.