Analysis

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These lecture notes draw on material found in Rudin (1976) and Stokey, Lucas and Prescott (1989).

1 Warm up: the vector space $\mathbb{R}^N$

1.1 $\mathbb{R}^N$ basics

Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{R}$ the set of real numbers. For any $N \in \mathbb{N}$, the N-dimensional real (Euclidean) space is the N-fold Cartesian product of $\mathbb{R}$. We denote this space by $\mathbb{R}^N$. Sometimes we abuse notation by letting N denote the set $(1, 2, \ldots, N)$ so that an element $x \in \mathbb{R}^N$ can be denoted by $(x_i)_{i\in N}$. It is often convenient to represent an element in $\mathbb{R}^N$ in vector form (as an $N \times 1$ column vector):

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

Associated with $\mathbb{R}^N$ are two basic operations:

(i) vector addition by $x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_N + y_N) \in \mathbb{R}^N$; and

(ii) scalar multiplication is for any scalar $\alpha \in \mathbb{R}$, defined by $\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_N) \in \mathbb{R}^N$.

$\mathbb{R}^N$, when combined with vector addition and scalar multiplication, is an example of a vector space. I define general vector spaces in Section 3.

1.2 Inner Product

Recall that a point $x \in \mathbb{R}^N$ can be represented as an $(N \times 1)$ column vector, so that its transpose x' is a $(1 \times N)$ row vector:

$$x' = \begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix}.$$

Definition 1. For any $x, y \in \mathbb{R}^N$, the *inner product* of x and y is given by:

$$x \cdot y = x'y = \sum_{i \in N} x_i y_i.$$

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1.3 The Euclidean Norm

Definition 2. The *Euclidean norm* of $x \in \mathbb{R}^N$, written $\|x\|$ is given by:

$$\|x\| = \sqrt{x \cdot x} = \left(\sum_{i \in N} x_i^2 \right)^{1/2}.$$

If $N = 1$, then the Euclidean norm is simply the absolute value of x: $\|x\| = |x|$.

The standard interpretation of the Euclidean norm is that it is a measure of the distance from the point to the origin. When $N = 1$, this is simply the absolute value of x.

When $N = 2$, consider an element $x = (x_1, x_2) \in \mathbb{R}^2$. To get the distance from this point to the origin, we can apply Pythagorean's theorem:

$$\|x\| = \sqrt{x_1^2 + x_2^2},$$

which coincides with the Euclidean norm.

2 Another warm-up: correspondences and functions

Definition 3. Let X and Y be two nonempty sets. A *correspondence* f from a set X into a set Y, denoted $f : X \to Y$ is a rule that assigns to each $x \in X$ a set $f(x) \subset Y$.

We are often interested in a particular type of correspondence where the set $f(x)$ is a singleton.

Definition 4. Let X and Y be two nonempty sets. A *function* f from a set X into a set Y, denoted $f : X \to Y$ is a rule that assigns to each $x \in X$ a *unique* $f(x) \subset Y$.

3 Metric Spaces and Normed Vector Spaces

Much of economics relies only on vectors in a finite-dimensional Euclidean space $\mathbb{R}^K$. But this is a prominent example of the much more extensive class of real linear spaces, some of them infinite-dimensional, which arise naturally when one considers difference and differential equations, optimal control problems, and dynamic programming.

Definition 5. A *real vector space* X, is a set of elements (vectors) together with two operations, addition and scalar multiplication, as well as a unique null (or zero) vector $\theta \in X$. For any two vectors $x, y \in X$, addition gives a vector $x + y \in X$; and for any vector $x \in X$ and any real number $\alpha \in \mathbb{R}$, scalar multiplication gives a vector $\alpha x \in X$.

These operations obey the usual algebraic laws; that is, for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$

(a) $x + y = y + x$

(b) $(x + y) + z = x + (y + z)$

(c) $\alpha(x + y) = \alpha x + \alpha y$

(d) $(\alpha + \beta)x = \alpha x + \beta x$;and
(e) $(\alpha\beta)x = \alpha(\beta x)$
Moreover the zero vector has the following properties:
(f) $x + \theta = x$; and
(g) $0x = \theta$.
Finally,
(h) $1x = x$.

Important features of a vector space are that it has a "zero" element and that it is closed under addition and scalar multiplication. Vector spaces are also called *linear spaces*. $\mathbb{R}^N$ is the canonical example of a vector space.

Exercise 1. Stokey, Lucas and Prescott (1989) Exercise 3.2.
Show that the following are vector spaces:
1. any finite-dimensional Euclidean space $\mathbb{R}^N$.
2. the set X consisting of all infinite sequences, $\{x_0, x_1, x_2, \ldots\}$ where $x_i \in \mathbb{R}$ for all $i = 0, 1, \ldots$ We call this set $\mathbb{R}^\omega$. *Note: you can define vector addition and scalar multiplication for $\mathbb{R}^\omega$ as we did for $\mathbb{R}^N$.*
3. the set of all continuous functions on the closed interval $[a, b] \subset \mathbb{R}$. We call this set $C[a, b]$. *Note: for $f, g \in C[a, b]$, define $h = f + g$ to be $h(x) = f(x) + g(x)$ for all $x \in [a, b]$. Similarly, for $f \in C[a, b]$, and $\alpha \in \mathbb{R}$, define $h = \alpha f$ to be $h(x) = \alpha f(x)$ for all $x \in [a, b]$. For this exercise, you can use the fact that if f, g are continuous then the sum $f + g$ is continuous, and the fact that the product αf is continuous for any $\alpha \in \mathbb{R}$. We will prove this later.*

Show that the following are not vector spaces:
4. the unit circle in $\mathbb{R}^2$.
5. the set of all integers, $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$.
6. the set of all non-negative functions on $[a, b] \subset \mathbb{R}$.

3.1 Metric Spaces

The vector space structure is not enough—on its own—to allow us to express certain concepts. In order to discuss convergence in a vector space or in any other space, we need to have a notion of distance.

Recall that, in $\mathbb{R}$ (i.e. $N = 1$), the distance between any two points $x, y \in \mathbb{R}$ is the absolute value:

$$d = |x - y|$$

and in $\mathbb{R}^2$, the Pythagorean theorem tells us that the distance between any two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ is the positive solution for d to the following equation:

$$d^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2.$$

More generally, in $\mathbb{R}^N$ the notion of distance is the Euclidean norm defined and described in Subsection 1.3.

But we may not always be in $\mathbb{R}^N$. In such a case, the notion of distance in Euclidean space can be generalized to the abstract notion of a *metric*, a function defined on any two elements that satisfies some basic conditions. We define a metric space as follows.

Definition 6. A *metric space* is a set X, together with a metric (a distance function) $\rho : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$
(a) $\rho(x, y) \geq 0$ and with equality if and only if $x = y$
(b) $\rho(x, y) = \rho(y, x)$ (symmetry), and
(c) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ (triangle inequality).

That is, a metric satisfies four basic properties: it is strictly positive between any two *distinct* points, equal to zero if and only if the two points are identical, it is symmetric, and it satisfies the triangle inequality.

Exercise 2. Stokey, Lucas and Prescott (1989) Exercise 3.3.
Show that the following are metric spaces:
1. the set of integers, $\mathbb{Z}$, with $\rho(x, y) = |x - y|$.
2. the set of integers, $\mathbb{Z}$, with $\rho(x, y) = 0$ if $x = y$ and $\rho(x, y) = 1$ if $x \neq y$.
3. the set of all continuous, strictly increasing functions on $[a, b]$ with

$$\rho(x, y) = \max_{t \in [a,b]} |x(t) - y(t)|.$$

4. the set $X = \mathbb{R}$, with $\rho(x, y) = f(|x - y|)$ where $f : \mathbb{R} \to \mathbb{R}$ is continuous, strictly increasing, and strictly concave, with $f(0) = 0$.

3.2 Normed Vector spaces

For vector spaces, metrics are usually defined in such a way that the distance between any two points is equal to the distance of their difference from the zero point. That is, since for any points x and y in a vector space X, the point $x - y$ is also in X, the metric on a vector space is usually defined in such a way that $\rho(x, y) = \rho(x - y, \theta)$. To define such a metric, we need the concept of a *norm*.

Definition 7. A *normed vector space* is a vector space X, together with a norm $\|\cdot\| : X \to \mathbb{R}$ such that for all $x, y \in X$ and $\alpha \in \mathbb{R}$
(a) $\|x\| \geq 0$ and with equality if and only if $x = \theta$
(b) $\|\alpha x\| = |\alpha| \cdot \|x\|$
(c) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

It is standard to view any normed vector space $(X, \|\cdot\|)$ as a metric space where the metric is taken to be $\rho(x, y) = \|x - y\|$ for all $x, y \in X$. Many, although not all, of the metric spaces used in economics are normed vector spaces.

Theorem 1. *Let $(X, \|\cdot\|)$ be a normed vector space. Define $\rho(x, y) = \|x - y\|$ for all $x, y \in X$. Then (X, ρ) is a metric space.*

Proof. We simply verify that the three metric properties hold (Definition 6).

(a) $\|x-y\| \geq 0$ and with equality iff $x-y=0$. This implies that $\rho(x,y) \geq 0$ and with equality iff $x=y$.

(b) $\|\alpha(x-y)\| = |\alpha| \cdot \|x-y\| = |-\alpha| \cdot \|x-y\| = \|\alpha(y-x)\|$. Setting $\alpha = 1$, we have that $\rho(x,y) = \rho(y,x)$.

(c) $\|x-y\| = \|x-z+z-y\| \leq \|x-z\| + \|z-y\|$. Therefore

$$\rho(x,y) \leq \rho(x,z) + \rho(z,y).$$

□

4 Examples of Normed Vector Spaces

4.1 Euclidean space

Theorem 2. *The* Euclidean space*, $X = \mathbb{R}^N$ with the Euclidean norm*

$$\|x\| = \left(\sum_{i \in N} x_i^2\right)^{1/2},$$

is a normed vector space.

In order to prove Theorem 2, we must show that the norm satisfies all three properties of Definition 7. The first two properties are trivial, while the third, the triangle inequality, is not. As an intermediate step, I first prove the following theorem, called the Cauchy-Schwartz Inequality, which is of independent interest.

Theorem 3. (Cauchy-Schwartz Inequality.) *If $x, y \in \mathbb{R}^N$ then*

$$|x \cdot y| \leq \|x\|\|y\|,$$

Moreover, for $x, y \neq \theta$, this weak inequality holds with equality if and only if x and y are colinear.

Proof. First, if $x \cdot y = 0$, then the inequality holds since both $\|x\| \geq 0$ and $\|y\| \geq 0$.

Next consider $x \cdot y \neq 0$. Note that this implies $x \neq \theta$, which further implies $\|x\| > 0$.

For any $\alpha \in \mathbb{R}$, since a sum of squares is always non-negative:

$$\begin{aligned} 0 &\leq (x-\alpha y) \cdot (x - \alpha y) \\ &= x \cdot x - 2\alpha(x \cdot y) + \alpha^2 (y \cdot y) \\ &= \|x\|^2 - 2\alpha(x \cdot y) + \alpha^2 \|y\|^2 \end{aligned}$$

Next set:

$$\alpha = \frac{\|x\|^2}{x \cdot y}$$

which is well defined since $x \cdot y \neq 0$. Then

$$0 \leq \|x\|^2 - 2\frac{\|x\|^2}{x \cdot y}(x \cdot y) + \frac{\|x\|^4}{(x \cdot y)^2}\|y\|^2$$

or

$$0 \leq \|x\|^2 (x \cdot y)^2 - 2\|x\|^2 (x \cdot y)^2 + \|x\|^4 \|y\|^2$$

Collecting terms and rearranging:

$$\|x\|^2 (x \cdot y)^2 \leq \|x\|^4 \|y\|^2$$

Since $\|x\| > 0$, one can divide both sides by $\|x\|^2$ and obtain:

$$(x \cdot y)^2 \leq \|x\|^2 \|y\|^2$$

And taking the square root of both sides yields

$$|x \cdot y| \leq \|x\| \|y\|.$$

Finally assume $x, y \neq \theta$. If x and y are colinear then there exists a $\lambda \in \mathbb{R}$ such that $x = \lambda y$. It is then easy to verify that the inequality holds with equality.

Conversely, if x and y are not colinear, then for any λ, $x - \lambda y \neq 0$. Hence $(x - \lambda y) \cdot (x - \lambda y) > 0$. Following the same argument as above with $\lambda = \|x\|^2/(x \cdot y)$ one can show that $|x \cdot y| < \|x\| \|y\|$. □

We now use the Cauchy Schwartz inequality in order to prove Theorem 2.

Proof. Proof of Theorem 2. (i) The first property is immediate:

$$\|x\| = \left(\sum_{i \in N} x_i^2 \right)^{1/2}$$

is positive and equal to zero if and only if $x = \theta$.

(ii) For any $\alpha \in \mathbb{R}$,

$$\|\alpha x\| = \left(\sum_{i \in N} \alpha^2 x_i^2 \right)^{1/2} = \left(\alpha^2\right)^{1/2} \|x\| = |\alpha| \cdot \|x\|$$

(iii) We have that:

$$\|x + y\| = \sqrt{(x + y) \cdot (x + y)}$$

Thus

$$\begin{aligned}\|x+y\|^2 &= (x+y)\cdot(x+y)\\ &= x\cdot x + 2x\cdot y + y\cdot y\\ &= \|x\|^2 + 2x\cdot y + \|y\|^2\end{aligned}$$

By the Cauchy-Schwartz inequality, $x\cdot y \le |x\cdot y| \le \|x\|\|y\|$. Then

$$\|x+y\|^2 \le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2.$$

Taking the square root of both sides yields the desired inequality. □

For $\mathbb{R}^N$, the Euclidean norm is the default norm. However, there are other possible norms for $\mathbb{R}^N$, infinitely many in fact. However, unless specified explicitly otherwise, the norm in $\mathbb{R}^N$ is understood to be the Euclidean norm.

In what follows, we consider a few more examples of normed vector spaces.

4.2 $\mathbb{R}^N$ with the max norm

Let $X = \mathbb{R}^N$ and consider the following norm:

$$\|x\|_{max} = \max_n |x_n|$$

This is called the max norm. For example if $x = (2, 4, -7) \in \mathbb{R}^3$, then $\|x\|_{max} = 7$.

Theorem 4. *The set $X = \mathbb{R}^N$ with the max norm is a normed vector space.*

Exercise 3. Prove Theorem 4.

4.3 ℓ^∞ with the sup norm

Following the typical notation, we express sequences as $(x_0, x_1, x_2, \ldots)$ or $\{x_n\}_{n=0}^\infty$

As in the first exercise, consider the set all infinite sequences in $\mathbb{R}$, $(x_0, x_1, x_2, \ldots)$ where $x_n \in \mathbb{R}$ for all $n = 0, 1, \ldots$ We call this set $\mathbb{R}^\omega$.

Note that such a sequence, $(x_0, x_1, x_2, \ldots)$, can be thought of as an element in the countably infinite Cartesian product of $\mathbb{R}$:

$$\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\cdots$$

Furthermore, as any element in $\mathbb{R}^\omega$ is a sequence, we call $\mathbb{R}^\omega$ a *sequence space*.

For $\mathbb{R}^\omega$, the natural analog of the max norm is the sup norm. A problem, however, is that the sup is not well defined (or is defined to be infinite) for certain elements of this set, such as $(1, 2, 3, 4, \ldots)$. For this reason, we restrict attention to the subset of $\mathbb{R}^\omega$ that consists of all bounded sequences.

For the following definition, it is useful to note that a sequence $(x_0, x_1, x_2, \ldots)$ is simply a set, but with more structure than a set: it has an explicit ordering of its elements. [Whereas, a set is completely defined by its elements alone.] I can thus define bounds of sets, and this definition will directly apply to sequences as well.[1]

Definition 9. Let X be a non-empty set of reals, $X \subset \mathbb{R}$. X is *bounded above* if there exists an $M \in \mathbb{R}$ such that $x \leq M$ for all $x \in X$; we call M an upper bound of X.

A set $X \subset \mathbb{R}$ is *bounded below* if there exists an $M \in \mathbb{R}$ such that $x \geq M$ for all $x \in X$; we call M a lower bound of X.

A set $X \subset \mathbb{R}$ is *bounded* if it is bounded both above and below.

We call the set of all bounded, infinite sequences in $\mathbb{R}$, ℓ^∞. Formally:

$$\ell^\infty = \{(x_0, x_1, x_2, \ldots) \in \mathbb{R}^\omega : \exists M \in \mathbb{R} \quad \text{s.t.} \quad |x_n| \leq M, \forall n = 0, 1, \ldots\}$$

Let us first establish that ℓ^∞ is a vector space. Recall that in Exercise 1 you have already shown that $\mathbb{R}^\omega$ is a vector space. While ℓ^∞ is a subspace of $\mathbb{R}^\omega$, it is not readily obvious that ℓ^∞ is also vector space—this requires showing explicitly that the vector space properties hold, i.e. that it is closed under both vector addition and scalar multiplication.

Exercise 4. Show that ℓ^∞ is a vector space as in Definition 5.

Next, having established that ℓ^∞ is a vector space, as we said the natural analog of the max norm for this space is the sup norm. It would behoove us to first define the least upper bound of a set.

[Note that in these lecture notes, for the sake of simplicity, I define upper bounds, lower bounds, $\sup$*, and* $\inf$ *for sets of reals. However, these definitions can clearly be generalized to sets in ordered spaces; see Rudin (1976).]*

Definition 10. Let X be a non-empty set of reals, $X \subset \mathbb{R}$, and X is bounded above. A real number $\alpha \in \mathbb{R}$ is the *least upper bound*, or the *supremum*, of X,

$$\alpha = \sup X.$$

if it satisfies the following properties: (i) α is an upper bound of X, and, (ii) if $\gamma < \alpha$, then γ is not an upper bound of X.

The *greatest lower bound*, or *infimum*, of a set $X \subset \mathbb{R}$ which is bounded below is defined in a similar manner:

$$\alpha = \inf X$$

if (i) α is an lower bound of X and (ii) if $\beta > \alpha$ then β is not an lower bound of X.

With this definition in hand, we state the following theorem.

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[1] To see this, here is the explicit definition of a bounded sequence in $\mathbb{R}$.

Definition 8. A sequence $(x_0, x_1, x_2, \ldots) \in \mathbb{R}^\omega$ is bounded above if there exists an $M \in \mathbb{R}$ such that $x_n \leq M$ for all $n = 0, 1, \ldots$; we call M an upper bound of x. A sequence $(x_0, x_1, x_2, \ldots) \in \mathbb{R}^\omega$ is bounded below if there exists an $M \in \mathbb{R}$ such that $x_n \geq M$ for all $n = 0, 1, \ldots$; we call M a lower bound of x. A sequence $(x_0, x_1, x_2, \ldots) \in \mathbb{R}^\omega$ is bounded if it is bounded both above and below.

Theorem 5. *ℓ^∞, the set of all bounded infinite sequences in $\mathbb{R}$, with the sup norm*

$$\|x\|_{\text{sup}} = \sup_n |x_n|,$$

is a normed vector space.

Exercise 5. Prove Theorem 5.

4.4 $C[a,b]$ with the sup norm

Let $[a,b]$ be a closed interval on $\mathbb{R}$. Let $C[a,b]$ be the set of all continuous functions on $[a,b]$. Because any element in $C[a,b]$ is a function, we call $C[a,b]$ a *function space.*

Recall that in Exercise 1 you have already shown that $C[a,b]$ is a vector space. For this, we defined vector addition and scalar multiplication in the following manner. For $f, g \in C[a,b]$, define $h = f + g$ to be $h(x) = f(x) + g(x)$ for all $x \in [a,b]$. Similarly, for $f \in C[a,b]$, and $\alpha \in \mathbb{R}$, define $h = \alpha f$ to be $h(x) = \alpha f(x)$ for all $x \in [a,b]$.

Similar to ℓ^∞, the natural norm on this space is the sup norm.

Theorem 6. *$C[a,b]$, the set of all continuous functions on $[a,b] \subset \mathbb{R}$, with the sup norm*

$$\|x\|_{\text{sup}} = \sup_{t \in [a,b]} |x(t)|,$$

is a normed vector space.

Exercise 6. Prove Theorem 6.

5 Sequences, Limits, Convergence, and Continuity in metric spaces

Now that we have a notion of distances, we can consider limits and convergence. The notion of convergence of a sequence of real numbers carries over without any change to a metric space (X, ρ).

5.1 Balls

Let (X, ρ) be a metric space.

Definition 11. For $\epsilon \in \mathbb{R}$, $\epsilon > 0$, the *open ϵ-ball* around x is

$$B_\epsilon(x) = \{a \in X : \rho(x,a) < \epsilon\}$$

We call this an “open ball” or a “neighborhood.”

As an example, consider Euclidean space. In $\mathbb{R}$ ($N = 1$), the open ball $B_\epsilon(x)$ is simply the interval $(x - \epsilon, x + \epsilon)$. In $\mathbb{R}^2$, $B_\epsilon(x)$ is the disk of radius ϵ centered at x, excluding the boundary circle. In $\mathbb{R}^3$, $B_\epsilon(x)$ is the solid ball of radius ϵ centered at x, excluding the boundary sphere.

5.2 Sequences and Convergence

We've already talked a bit about infinite sequences in $\mathbb{R}$. For example, we've called $\mathbb{R}^\omega$ the set of all infinite sequences, $(x_0, x_1, x_2, \ldots)$ where $x_n \in \mathbb{R}$ for all $n = 0, 1, \ldots$But now we would like to consider sequences in more general metric spaces.

Let (X, ρ) be a metric space. We will now think about infinite sequences, $(x_0, x_1, x_2, \ldots)$ or $\{x_n\}_{n=0}^{\infty}$, where $x_n \in X$ for all $n = 0, 1, \ldots$

Note that one can think of a sequence as a function mapping from the natural numbers (and, in my notation, zero) to X.

Definition 12. A sequence $\{x_n\}_{n=0}^{\infty}$ in X *converges* to a *limit point* $x \in X$, also written

$$x_n \to x \in X,$$

if for each $\epsilon > 0$ there exists an $N_\epsilon \in \mathbb{N}$ such that for all $n \geq N_\epsilon$

$$\rho(x_n, x) < \epsilon.$$

[Equivalently, for all $n \geq N_\epsilon$, $x_n \in B_\epsilon(x)$.]

In other words, $x_n \to x$ if and only if the sequence of distances for x, $\{\rho(x_n, x)\}_{n=0}^{\infty}$, which itself is a sequence in $\mathbb{R}_+$, converges to zero.

In a metric space, a sequence cannot converge to two different points.

Theorem 7. *Let (X, ρ) be a metric space and $\{x_n\}$ a sequence in X. If $x_n \to x$ and $x_n \to y$ then $x = y$. That is, if $\{x_n\}$ has a limit, then that limit is unique*

Proof. Fix any $\epsilon > 0$. If $x_n \to x$, then there is a N_ϵ^x such that for all $n \geq N_\epsilon^x$, $\rho(x_n, x) < \epsilon/2$. Similarly, if $x_n \to y$, then there is a N_ϵ^y such that for all $n \geq N_\epsilon^y$, $\rho(x_n, y) < \epsilon/2$. Therefore, by the triangle inequality, for any $n \geq \max\{N_\epsilon^x, N_\epsilon^y\}$,

$$\rho(x, y) \leq \rho(x, x_n) + \rho(x_n, y) < \epsilon.$$

Since ϵ was arbitrary, this implies, this implies $\rho(x, y) = 0$, hence $x = y$. $\square$

5.3 Subsequences and Convergence

Given a sequence, one can construct a new sequence, called a subsequence, out of the original sequence. [For this definition, I'm going to slightly change notation so that sequences start with index $n = 1$ rather than zero. This is without loss, it just makes the definition a bit less clunky!]

Definition 13. Given a sequence $\{x_n\}_{n=1}^{\infty}$ in X, a sequence $\{a_k\}_{k=1}^{\infty}$ in X is a *subsequence* of $\{x_n\}$ if there exists a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$, such that for all $k \in \mathbb{N}$:

$$n_k \in \mathbb{N} \qquad \text{and} \qquad a_k = x_{n_k}.$$

That is, a subsequence is a selection of some (possibly all) members of the original sequence that preserves the original order. It is easy to see that even if a sequence $\{x_n\}$ fails to converge, some of its subsequences may converge.

As an example, take the sequence in $\mathbb{R}$ of alternating ones and zeros:

$$\{x_n\} = (1, 0, 1, 0, \ldots).$$

This sequence does not converge. However, from this sequence we can form the following subsequence:

$$(1, 1, 1, \ldots).$$

This subsequence converges to 1. There are, of course, other subsequences of $\{x_n\}$ that do not converge, including the original sequence itself.

Theorem 8. *Let (X, ρ) be a metric space. A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if every subsequence of $\{x_n\}$ converges to $x \in X$.*

Exercise 7. Prove Theorem 8.

5.4 Cauchy Sequences and Completeness

Verifying convergence involves having a "candidate" limit point x. However, in many applications we may not know the limit. If such a candidate is unknown, the following criterion is useful.

Definition 14. A sequence $\{x_n\}$ in X is a *Cauchy sequence*, or satisfies the *Cauchy criterion*, if for each $\epsilon > 0$ there exists an $N_\epsilon \in \mathbb{N}$ such that

$$\rho(x_n, x_m) < \epsilon, \qquad \forall n, m \geq N_\epsilon.$$

A sequence is Cauchy if the elements of the sequence get closer and closer to each other. We next establish some basic facts about the Cauchy criterion.

Theorem 9. *Let (X, ρ) be a metric space and $\{x_n\}$ a sequence in X. If $\{x_n\}$ converges, then it is Cauchy.*

Proof. Suppose $x_n \to x$ and fix any $\epsilon > 0$. Then there exists is an N_ϵ such that for all $n \geq N_\epsilon$,

$$\rho(x_n, x) < \epsilon/2.$$

Then for any $n, m \geq N_\epsilon$ the following holds by the triangle inequality:

$$\rho(x_n, x_m) \leq \rho(x_n, x) + \rho(x, x_m) < \epsilon.$$

□

The following fact about Cauchy sequences I leave as an exercise.

Exercise 8. Let $\{x_n\}$ be a sequence in $\mathbb{R}$. That is, $\{x_n\} \in \mathbb{R}^\omega$. Show that if $\{x_n\}$ satisfies the Cauchy criterion, then it is bounded.

Completeness. We have established in Theorem 9 that any sequence in X that converges is Cauchy. If the converse is always true, if Cauchy sequences always converge, then we say the set is complete.

Definition 15. A metric space (X, ρ) is *complete* if every Cauchy sequence in X converges to an element in X.

Complete metric spaces are useful for us in the following sense. If a metric space is known to be complete, you can establish that a sequence in X converges to a limit point in X (i.e. establish the existence of a limit), simply by verifying that the sequence satisfies the Cauchy criterion. This is a great advantage, as checking the Cauchy criterion can be readily accomplished with only the knowledge of the sequence itself, and without the knowledge of the limit point!

Establishing whether a particular metric space is complete takes some work. We will take the following as given.

Theorem 10. *The Euclidean space, $X = \mathbb{R}^N$, with the Euclidean norm is a complete metric space.*

Proof. Omitted. See Rudin (1976), pages 53-55. □

Understanding the proof of this statement is not necessary for what follows in this course and for this reason I omit it. However, the basic fact that the Euclidean space is complete will be relied upon!

Definition 16. A complete normed vector space is called a *Banach space*.

5.5 Continuity

Throughout I have assumed that you have at the very least an informal understanding of continuity. Given the preceding discussion of limits, it is now relatively easy to formally define continuity.

Definition 17. Let (X, ρ_X) and (Y, ρ_Y) be a metric space. A function $f : X \to Y$ is *continuous at* $\bar{x} \in X$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\rho_Y(f(x), f(\bar{x})) < \epsilon$$

for all $x \in X$ for which

$$\rho_X(x, \bar{x}) < \delta.$$

[equivalently, for all $x \in B_\delta(\bar{x})$.]

A function f is *continuous* if it is continuous at all $\bar{x} \in X$.

Intuitively, as x gets closer to $\bar{x}$, $f(x)$ gets closer to $f(\bar{x})$.

The above definition is the general definition of continuity, e.g. as found in Rudin (1976). It defines continuity for functions that map from one metric space into another.

Again, for the sake of simplicity, I'm going to rewrite this definition in terms of functions that map from $\mathbb{R}^N$ to $\mathbb{R}$ (both with the Euclidean norm). I do this because these are the typical types of functions we work with in economics.

Definition 18. Let $X \subseteq \mathbb{R}^N$ with the Euclidean norm, $\|\cdot\|$. A function $f : X \to \mathbb{R}$ is *continuous at* $\bar{x} \in X$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(\bar{x})| < \epsilon$$

for all $x \in X$ for which

$$\|x - \bar{x}\| < \delta.$$

Exercise 9. Let (X, ρ) be a metric space and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be continuous. Let $h : X \to \mathbb{R}$ be the function defined by $h(x) = f(x) + g(x)$ for all $x \in X$.

Prove that h is continuous. That is, the sum of two continuous functions is continuous.

5.6 An Example of a Complete Metric Space

I will illustrate verifying completeness with an example that will be important for what follows in this class.

Theorem 11. *Let $X \subseteq \mathbb{R}^N$. Let $C(X)$ be the set of bounded, continuous functions*

$$f : X \to \mathbb{R}$$

with the sup norm

$$\|f\| = \sup_{x \in X} |f(x)|.$$

Then $C(X)$ is a complete, normed vector space.

Proof. That $C(X)$ is a normed vector space follows from Exercise 6. Hence it suffices to show that if $\{f_n\}$ is a Cauchy sequence in $C(X)$, then it converges. Let me state this more clearly. We want to prove the following statement:

Consider a sequence $\{f_n\}$ that is Cauchy. That is, for each $\epsilon > 0$ there exists an $N_\epsilon \in \mathbb{N}$ such that

$$\rho(f_n, f_m) = \|f_n - f_m\| < \epsilon, \qquad \forall n, m \geq N_\epsilon.$$

Then there exists an $f \in C(X)$ such that for any $\epsilon > 0$, there exists an $N'_\epsilon \in \mathbb{N}$ such that

$$\rho(f_n, f) = \|f_n - f\| < \epsilon, \qquad \forall n \geq N'_\epsilon.$$

Three steps are involved in this proof: (i) we first find a "candidate" limit function f, (ii) we then show that $\{f_n\}$ converges to f in the sup norm, and (iii) we show that $f \in C(X)$, i.e. that f is bounded and continuous.

Part (i). Fix an element $x \in X \subseteq \mathbb{R}^N$. Then the sequence of real numbers $\{f_n(x)\} \in \mathbb{R}^\omega$ satisfies

$$|f_n(x) - f_m(x)| \leq \sup_{y \in X} |f_n(y) - f_m(y)| = \|f_n - f_m\|.$$

Therefore, the sequence $\{f_n(x)\}$ satisfies the Cauchy criterion. By the completeness of the real numbers [Theorem 10], it converges to a limit point. Call this limit point $f(x)$.

Our candidate limit function f, then, will be the function $f : X \to \mathbb{R}$ defined by this process for all $x \in X$.

Part (ii). Next we must show that $\|f_n - f\| \to 0$. Fix an arbitrary $\epsilon > 0$. Given that $\{f_n\}$ is a Cauchy, we know that there exists an $N_{\epsilon/2} \in \mathbb{N}$ such that

$$\rho(f_n, f_m) = \|f_n - f_m\| < \epsilon/2, \qquad \forall n, m \geq N_{\epsilon/2}.$$

Now for any fixed $x \in X$ and all $m \geq n \geq N_{\epsilon/2}$:

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\ &\leq \|f_n - f_m\| + |f_m(x) - f(x)| \\ &< \epsilon/2 + |f_m(x) - f(x)| \end{aligned}$$

where the first line follows from the triangle inequality, the second from the definition of the sup norm, and the third from the fact that $\{f_n\}$ is a Cauchy.

Recall that in part (i) we have established that the sequence $\{f_m(x)\}$ converges to the limit point $f(x)$. This implies that given $\epsilon/2 > 0$, there exists an $M_{\epsilon/2} \in \mathbb{N}$ such that for all $m \geq M_{\epsilon/2}$

$$|f_m(x) - f(x)| < \epsilon/2.$$

Therefore, we can just choose an $m \geq M_{\epsilon/2}$, and the following is true:

$$|f_n(x) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since the choice of x was arbitrary, it follows that

$$\|f_n - f\| < \epsilon, \qquad \forall n \geq N_{\epsilon/2}.$$

Since $\epsilon > 0$ was arbitrary, it follows that $\|f_n - f\| \to 0$.

Part (iii). Finally, we must show that $f \in C(X)$, i.e. that it is bounded and continuous. Boundedness is obvious, but continuity is not.

To prove that f is continuous, we must show that for every $\epsilon > 0$ and every $\bar{x} \in X$, there exists a $\delta > 0$ such that

$$|f(x) - f(\bar{x})| < \epsilon \qquad \text{if} \qquad \|x - \bar{x}\|_E < \delta$$

where $\|\cdot\|_E$ denotes the Euclidean norm on $\mathbb{R}^N$.

Let $\epsilon > 0$ and $\bar{x} \in X$ be given. Choose k so that

$$\|f - f_k\| < \epsilon/3.$$

Since $f_n - f$ in the sup norm, such a choice is possible.

Next, choose δ so that

$$\|x - \bar{x}\|_E < \delta \qquad \text{implies} \qquad |f_k(x) - f_k(\bar{x})| < \epsilon/3.$$

Since f_k is a continuous function, such a choice is possible.

Then

$$\begin{aligned} |f(x) - f(\bar{x})| &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(\bar{x})| + |f_k(\bar{x}) - f(\bar{x})| \\ &\leq 2\|f - f_k\| + |f_k(x) - f_k(\bar{x})| \end{aligned}$$

where the first line follows from the triangle inequality, and the second by the definition of the sup norm. Finally, this implies

$$|f(x) - f(\bar{x})| < \epsilon.$$

Given that our choice of $\bar{x} \in X$ was arbitrary, f is continuous. Therefore $f \in C(X)$. $\square$

Note that convergence of a sequence of functions $\{f_n\}$ in the sup norm is the same as uniform convergence.

References

Rudin, Walter, *Principles of Mathematical Analysis*, 3rd ed., McGraw-Hill, New York, NY, 1976.

Stokey, Nancy L., Robert E. Lucas, and Edward C. Prescott, *Recursive Methods in Economic Dynamics*, Harvard University Press, 1989.