Incomplete Markets and Self-Insurance

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We now relax our very strong assumption of complete markets and consider models in which agents face incomplete insurance markets. We will consider a savings problem of an individual household facing an uncertain income stream. However, instead of being able to trade in a complete Arrow-Debreu market of securities, this household is cut off from all insurance markets and can only purchase non-negative amounts of a single risk-free asset. The absence of insurance induces the household to adjust its asset holdings in order to "self-insure" against income shocks.

This lecture builds on material found in Chapters 16 and 17 of Ljungqvist and Sargent (2004).

1 The Individual Household's Problem

The household has the following preferences over consumption

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t u(c_t)$$

with $\beta \in (0,1)$. We assume $u(c)$ satisfies the typical regularity conditions: it is continuous, strictly increasing, strictly concave, continuously differentiable.

The agent is endowed with an infinite random sequence $\{y_t\}_{t=0}^{\infty}$ of the consumption good. Each period, the endowment takes one of a finite number of values, indexed by $s \in S$. In particular the set of possible endowments is

$$y \in \mathcal{Y} = \{\bar{y}_1, \bar{y}_2, \dots \bar{y}_S\}$$

where without loss of generality we assume

$$\bar{y}_1 < \bar{y}_2 < \dots < \bar{y}_S$$

We assume that the sequence of endowments are i.i.d. across time and drawn from the following distribution

$$\Pr(y_t = \bar{y}_s) = \pi_s$$

with $\sum \pi_s = 1$. Finally, there are no insurance markets.

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We assume that the household can hold non-negative amounts of a single risk-free asset that has an exogenous net rate of return r. Throughout this lecture we assume $(1+r)\beta = 1$. Note that this interest rate is not endogenously determined as it has been in previous lectures, it is just assumed to be equal to the discount rate. That is, we are considering a *partial* equilibrium model. We will relax this assumption later.

The agent faces the sequence of budget constraints

$$c_t + b_t = y_t + \frac{1}{1+r} b_{t+1} \tag{1}$$

where b_t is the total amount of debt the consumer has at the beginning of period t (debt brought into period t). The left hand side is the consumer's consumption expenditures and its repayment of debt. The right hand side is the consumer's current realization of income and new issuances of debt.

An alternative way of writing the household's per-period budget constraint is to let a_t denote the assets of the consumer at the beginning of in period t *including* the current realization of the income process. That is, let:

$$a_t = -b_t + y_t$$

Substituting this into (1) we get

$$\begin{aligned} c_t + (y_t - a_t) &= y_t + \frac{1}{1+r}(y_{t+1} - a_{t+1}) \\ c_t - a_t &= \frac{1}{1+r}(y_{t+1} - a_{t+1}) \end{aligned}$$

Therefore we may alternatively write the agent's per-period budget constraint as

$$a_{t+1} = (1+r)(a_t - c_t) + y_{t+1}. \tag{2}$$

We will switch back and forth between these conventions

We impose the constraint that assets must be non-negative by the end of every period: $a_t - c_t \geq 0$. That is,

$$0 \leq c_t \leq a_t$$

The constraint $c_t \geq 0$ is either imposed or comes from an Inada condition that $\lim_{c \to 0} u'(c) = \infty$.

Given the consumer's problem, the Bellman equation for the agent is given by

$$V(a) = \max_{c \in [0,a]} \left\{ u(c) + \beta \sum_{s \in S} \pi_s V((1+r)(a-c) + \bar{y}_s) \right\}$$

One can show that the value function $V(a)$ inherits the basic properties of $u(c)$, that is, it is increasing, strictly concave, and differentiable.

2 Nonstochastic Endowment

While in the environment described above the household faced an uncertain income stream, we first consider an environment in which there is no uncertainty over income. That is, y_t may vary over time, but all movements in income are known at time $t = 0$. Without uncertainty, the question of insurance becomes moot.

For now let's use our debt notation, where b_t is the consumer's debt brought into period t. With this notation, the time t budget constraint is given by:

$$c_t + b_t = y_t + \frac{1}{1+r} b_{t+1}. \tag{3}$$

We will impose a no-borrowing constraint that debt cannot be positive, it can only be weakly negative

$$b_{t+1} \leq 0$$

That is, the household cannot borrow—it can only save. This no-borrowing constraint is more stringent than the natural borrowing constraint when there are Arrow-Debreu securities, or equivalently, one-period Arrow securities. Let us first consider this case.

2.1 The Natural Borrowing Constraint

We first solve for the optimal consumption path under what is called the natural borrowing constraint. The natural borrowing constraint in this economy is given by iterating (3) forward. Imposing the condition that $c_t \geq 0$, this gives us

$$b_t \leq \bar{b}_t \equiv \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_{t+j} \tag{4}$$

This is the maximal amount that the agent can borrow: the greatest amount feasible it can repay based on future income.

Iteration on the budget constraint (3) and imposing the initial condition that $b_0 = 0$, we obtain:

$$\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t c_t \leq \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t y_t. \tag{5}$$

Therefore, the present discounted value of lifetime consumption is equal to the present discounted value of lifetime income. Under the natural borrowing constraint, this is the only restriction that the budget constraints impose on the sequence of consumptio $\{c_t\}_{t=0}^{\infty}$, and this is exactly the same single budget constraint we derived when markets were complete!

Intertemporal Budget Constraint with Arrow-Debrew securities. Pick an arbitrary scalar $q_0 > 0$ and define q_t recursively by

$$q_t = \frac{q_{t-1}}{1+r} = ... = \frac{q_0}{(1+r)^t}.$$

Note that q_t/q_s represents the price of the period t consumption relative to period s consumption. Without loss of generality, we normalize $q_0 = 1$ so that $q_t = q_t/q_0$. Then:

$$q_t = \frac{1}{(1+r)^t}$$

is the price of consumption at time t in terms of period 0 consumption.

With Arrow-Debreu securities, the household's intertemporal budget constraint is given by

$$\sum_{t=0}^{\infty} q_t c_t \leq \sum_{t=0}^{\infty} q_t y_t$$

Therefore in the setting without uncertainty, the budget constraint with AD securities is equivalent to the households intertemporal budget constraint with a natural borrowing limit, (5).

The household's problem. Consider now the household's problem of maximizing utility subject to (5). The first order conditions give us that

$$u'(c_t) = \beta(1+r)u'(c_{t+1})$$

as long as the natural borrowing constraint is slack. This can equivalently be written as:

$$\frac{u'(c_t)}{q_t} = \beta \frac{u'(c_{t+1})}{q_{t+1}}$$

Using the fact that $\beta(1+r) = 1$, we have that

$$c_t = c_{t+1} = \bar{c}, \qquad \forall t$$

[Parenthesis: Note that if instead $\beta(1+r) > 1$, this would imply $u'(c_t) > u'(c_{t+1})$ so that $c_t < c_{t+1}$; therefore consumption would grow over time. If on the other hand $\beta(1+r) < 1$, this would imply $u'(c_t) < u'(c_{t+1})$ so that $c_t > c_{t+1}$; therefore consumption would shrink over time.]

Substituting this solution into the budget constraint (5), we get that

$$\frac{1+r}{r}\bar{c} = \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t y_t$$

Therefore, consumption is given by a constant consumption level $c_t = \bar{c}$ given by

$$\bar{c} = \frac{r}{1+r} x_0, \qquad \forall t. \tag{6}$$

where

$$x_0 \equiv \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t y_t = \sum_{t=0}^{\infty} q_t y_t$$

Therefore, we get perfect consumption smoothing over time: the household's consumption path is completely flat. In particular, the household consumes the annuity value of its wealth. We x_0 the household's wealth or "permanent income" at time 0: it is the present discounted value of lifetime income. The household consumes the annuity value $r/(1+r)$ of its wealth or permanent income in every period, resulting in a perfectly flat consumption path.

Next we check whether under the optimal policy of the household the natural borrowing constraint ever binds. First let's solve for the household's debt level at any period. We have that b_t is given by

$$b_t = (1+r)(c_{t-1} + b_{t-1} - y_{t-1})$$

Now iterating this constraint backward we get

$$b_t = (1+r)(c_{t-1} - y_{t-1}) + (1+r)^2(c_{t-2} - y_{t-2}) + (1+r)^3(c_{t-3} - y_{t-4}) + \cdots$$

That is,

$$\begin{aligned} b_t &= \sum_{j=0}^{t-1}(1+r)^{t-j}(c_j - y_j) \\ &= (1+r)^t \sum_{j=0}^{t-1} \left(\frac{1}{1+r}\right)^j (c_j - y_j) \end{aligned}$$

Plugging in the fact that consumption is constant every period, we get that:

$$\begin{aligned} b_t &= (1+r)^t \sum_{j=0}^{t-1} \left(\frac{1}{1+r}\right)^j (\bar{c} - y_j) \\ &= (1+r)^t \left(\sum_{j=0}^{t-1} \left(\frac{1}{1+r}\right)^j \bar{c} - \sum_{j=0}^{t-1} \left(\frac{1}{1+r}\right)^j y_j \right) \\ &= (1+r)^t \left(\frac{1+r}{r}\bar{c} - \left(\frac{1}{1+r}\right)^t \frac{1+r}{r}\bar{c} - \sum_{j=0}^{t-1} \left(\frac{1}{1+r}\right)^j y_j \right) \\ &= (1+r)^t \frac{1+r}{r}\bar{c} - (1+r)^t \sum_{j=0}^{t-1} \left(\frac{1}{1+r}\right)^j y_j - \frac{1+r}{r}\bar{c} \end{aligned}$$

Finally, using the fact that the optimal consumption is given by the annuity value of permanent income (6), we get that debt at time t is given by

$$\begin{aligned} b_t &= (1+r)^t \left[\sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_j - \sum_{j=0}^{t-1} \left(\frac{1}{1+r}\right)^j y_j \right] - \frac{1+r}{r}\bar{c} \\ &= (1+r)^t \left[\sum_{i=t}^{\infty} \left(\frac{1}{1+r}\right)^i y_i \right] - \frac{1+r}{r}\bar{c} \end{aligned}$$

which we may rewrite (by changing the index $i = t + j$) as

$$b_t = \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j y_{t+j} - \frac{1+r}{r} \bar{c}$$

Therefore, debt is equal to the present discounted value of all *future* income minus the present discounted value of all future consumption.

This expression for debt b_t is obviously less than or equal to the natural debt limit $\bar{b}_t$ for all $t \geq 0$, given in (4) because $\bar{c}$ is strictly positive. Therefore, the natural debt limit never binds under the optimal consumption path. This is intuitive: the household would never find it optimal to choose a path for consumption which entails zero consumption for the rest of time starting at some period t.

Note that the natural debt limits allow b_t to be positive, but not too large. The agent takes on debt if the agent's income is growing, as the agent would like to borrow against future income in order to consume. That is, the agent optimally decides to shift future consumption to the present.

2.2 Ad hoc no borrowing constraint

We now go back to the original case in which the agent faces the much more severe ad hoc no borrowing constraint. We continue to assume that the endowment sequence is known so that there is no uncertainty, but now we impose the following constraint

$$b_{t+1} \leq 0, \qquad \forall t$$

This implies that the houeshold can save but not borrow. Let us now work with the asset notation as we described above, where

$$a_{t+1} = -b_{t+1} + y_{t+1}$$

That is, a_{t+1} are the assets the agents has at the beginning of time $t+1$ that includes the current realization of income. If the agent cannot enter the next period with debt, this just means that the borrowing constraint becomes

$$c_t \leq a_t$$

Then agent can only consume up to his asset holdings (which includes today's income).

Let (c_t^*, a_t^*) denote an optimal path of the agent. The household's problem is to maximize utility over consumption given the sequence of budget constraints

$$a_{t+1} = (1+r)(a_t - c_t) + y_{t+1} \tag{7}$$

and the no-borrowing constraint $c_t \leq a_t$. Let λ_t be the Lagrange multiplier on the no-borrowing constraint. The agent's first-order necessary conditions are

$$u'(c_t) = \beta(1+r)u'(c_{t+1}) + \lambda_t$$

where λ_t is strictly positive if and only if the borrowing constraint is binding. Therefore, using the fact that $\beta(1+r) = 1$ we get that:

$$u'(c_t^*) \geq u'(c_{t+1}^*)$$

This holds with equality if the borrowing constraint is not binding ($\lambda_t = 0$). Therefore, along an optimal path:

$$c_t^* = c_{t+1}^*, \qquad \text{when} \qquad c_t^* < a_t^*$$

and

$$c_t^* < c_{t+1}^*, \qquad \text{when} \qquad c_t^* = a_t^*.$$

This states that the no-borrowing constraint binds *only when* the household desires to shift consumption from the future to the present. The household will desire to do that only when its income is growing. Furthermore, according to these conditions, c_t can never exceed c_{t+1}.

Solving the budget constraint (7) forward for a_t and rearranging gives us

$$\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j c_{t+j} = a_t + \sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^j y_{t+j} \tag{8}$$

This holds for all dates $t \geq 1$ in which the consumer arrives with a strictly positive net asset position $a_t > y_t$. If instead the borrowing constraint was binding at time $t-1$, we have that $a_t = y_t$, and:

$$\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j c_{t+j} = \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j y_{t+j} \tag{9}$$

This implies that if the no-borrowing constraint is binding only finitely often, then after the last date at which the constraint was binding, (9) and the Euler equation together imply that consumption will thereafter be constant at the level $\bar{c}'$ that satisfies

$$\bar{c}' = \frac{r}{1+r}\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j y_{t+j}$$

This is similar to the consumption in the natural borrowing constraint economy, except this holds after the last date in which the constraint is binding. That is, suppose that an household arrives in period t with zero savings but knows that the borrowing constraint will never bind again. The household would then find it optimal to choose the highest sustainable constant consumption. This is again given by the annuity value of the tail end of the income process starting from period t. That is:

$$\bar{c}' = \frac{r}{1+r}x_t$$

where

$$x_t \equiv \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j y_{t+j}$$

is the present discounted value in period t of the tail end of the income process.

Thus, the impact of the borrowing constraint will not vanish until the date at which the annuity value of the tail of the income process is maximized. (Because the borrowing constraint only binds when the household wants to move consumption in the future to today.) We state this in the following proposition.

Proposition 1. *Given a borrowing constraint and a nonstochastic endowment stream, the limit of the nondecreasing optimal consumption path is*

$$\bar{c} \equiv \lim_{t\to\infty} c_t^* = \frac{r}{1+r}\bar{x}$$

where

$$\bar{x} \equiv \sup_t x_t$$

Proof. See proof in Chapter 16 of Ljungqvist and Sargent (2004). □

More generally, we know that at each date t for which the no-borrowing constraint is binding at date $t-1$, consumption will increase to satisfy (9). The time series of consumption will thus be a discrete time step function whose jump dates $\bar{t}$ coincide with the dates in which x_t attains new highs:

$$\bar{t} = \{t | x_t > x_s, \text{for all } s < t\}$$

Thus if there is a finite last date $\bar{t}$, optimal consumption is a monotone bounded sequence that converges to a finite limit.

3 The Permanent Income Hypothesis (PIH) with Quadratic Preferences

We now go back to allowing for uncertainty in the income process. We allow $\{y_t\}_{t=0}^{\infty}$ to be an arbitrary stationary stochastic process.

We consider the special case in which utility is quadratic:

$$u(c) = -\frac{1}{2}(c_t - \gamma)^2$$

where $\gamma > 0$ is called the bliss consumption level. This is useful because then marginal utility is linear in consumption

$$u'(c) = \gamma - c$$

Furthermore, we put no bounds on c in this case: consumption is allowed to be negative. Finally, we impose that

$$\mathbb{E}_0 \lim_{t\to\infty} \left(\frac{1}{1+r}\right)^t b_t^2 = 0$$

This constrains the asymptotic rate at which debt can grow.

The household's FOC is given by

$$u'(c) = \beta(1+r)\mathbb{E}_t u'(c_{t+1})$$

Next using the fact that $\beta(1+r) = 1$, we get:

$$u'(c) = \mathbb{E}_t u'(c_{t+1})$$

or,

$$\gamma - c_t = \mathbb{E}_t (\gamma - c_{t+1})$$

Therefore under linear marginal utility we get that

$$\mathbb{E}_t c_{t+1} = c_t$$

Therefore c_t follows a random walk!

Iterating the agent's budget constraint forward, we get that

$$b_t = \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j (y_{t+j} - c_{t+j}) \tag{10}$$

Taking expectations of both sides, we have:

$$\mathbb{E}_t b_t = \mathbb{E}_t \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j (y_{t+j} - c_{t+j}),$$

and note that $b_t = \mathbb{E}_t b_t$. Combining the optimality condition $\mathbb{E}_t c_{t+1} = c_t$ with the household's budget constraint, we get that

$$\begin{aligned}
b_t &= \mathbb{E}_t \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j (y_{t+j} - c_{t+j}) = \mathbb{E}_t \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_{t+j} - \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j \mathbb{E}_t c_{t+j} \\
&= \mathbb{E}_t \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_{t+j} - \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j c_t \\
&= \mathbb{E}_t \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_{t+j} - \frac{1+r}{r} c_t
\end{aligned}$$

Solving this for c_t, we obtain:

$$c_t = \frac{r}{1+r} \left[-b_t + \mathbb{E}_t \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_{t+j} \right] \tag{11}$$

This is what is called the Permanent Income Hypothesis (Friedman, 1957). Let

$$x_t = \mathbb{E}_t \left[\sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j y_{t+j} \right] - b_t$$

denote the agent's expected permanent income as of time t. Then equation (11) tells us that the household sets its current consumption equal to the annuity value of its expected permanent income.

$$c_t = \frac{r}{1+r} x_t$$

We say that the household's marginal propensity to consume from permanent income is $r/(1+r)$.

This consumption rule has the feature of certainty equivalence. That is, consumption c_t depends only on the first moment of the discounted value of the endowment sequence; consumption does not depend on any higher orders (like variance for example).

4 Estimation and Tests of the PIH

Next we ask: can we test this model? There are two main tests (generally done on aggregate data, but also people have looked it in micro data).

Random Walk Hypothesis and Unpredictability of Innovations in Consumption. Let's first look at innovations to consumption. Let

$$\Delta c_t \equiv c_t - c_{t-1}$$

denote the innovation in consumption between time t and $t-1$.

The fact that consumption follows a random walk implies that innovations to consumption are unpredictable at time $t-1$

$$\mathbb{E}_{t-1}[\Delta c_t] = 0$$

Thus, the first main test is this random walk result, the unpredictability of consumption innovations.

We can say more about these innovations. Let us rewrite the sequence budget constraint (1) as follows

$$b_t = -(1+r)(y_{t-1} - c_{t-1} - b_{t-1})$$

Substituting this into our expression for optimal consumption (11) we get

$$c_t = r(y_{t-1} - c_{t-1} - b_{t-1}) + \frac{r}{1+r} \mathbb{E}_t \left[\sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j y_{t+j} \right] \tag{12}$$

Next, lagging the optimal consumption function (11) by one period and multiplying through by

$(1+r)$ gives us

$$(1+r)c_{t-1} = r\left\{-b_{t-1} + y_{t-1} + \frac{1}{1+r}\mathbb{E}_{t-1}\left[\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j y_{t+j}\right]\right\} \tag{13}$$

Subtracting (13) from (12) gives us

$$c_t - c_{t-1} = \frac{r}{1+r}\mathbb{E}_t\left[\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j y_{t+j}\right] - \frac{r}{1+r}\mathbb{E}_{t-1}\left[\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j y_{t+j}\right]$$

Therefore the innovations to consumption satisfy

$$\Delta c_t = \frac{r}{1+r}\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j [\mathbb{E}_t y_{t+j} - \mathbb{E}_{t-1} y_{t+j}] \tag{14}$$

That is, the change in consumption from $t-1$ to t, which is unpredictable at time $t-1$, is related to "news" about income and future income. New information at time t will generally cause the consumer to revise previously held expectations about current and future income, so that the discounted present value of these expectations will itself change. Thus, innovations in consumption come from revisions in expectations of permanent income.

The unpredictability of aggregate consumption growth was first tested in a seminal paper by Hall (1978). This model states that variables lagged $t-1$ or earlier, in particular lags of income, should not help predict the change in consumption in period t. Hall found that lagged income terms were not significant in this regression. That is, he found orthogonality of lagged income to consumption changes, confirming the predictions of the permanent income hypothesis. He did however reject the model based on the finding that lagged stock-market values predicted consumption change.

Forming expectations about income. Suppose that we accept that changes in aggregate consumption cannot be predicted by lags of income. However, we can say more. According to (14), changes in consumption come from innovations in *expectations* about future labor income. Thus, if we know something about the underlying stochastic income generating process, we ought to be able to check this prediction too.

Suppose we assume $\{y_t\}_{t=0}^{\infty}$ is an MA(2) process.

$$y_t = \varepsilon_t + \lambda\varepsilon_{t-1}$$

where ε_t is an i.i.d. white noise process, typically assumed to be Gaussian: $\varepsilon_t \sim N(0,1)$. Thus, good news now not only revises current period's income upward, but also revises expectations of next period's income upward (when $\lambda > 0$, and downward when $\lambda < 0$). In this case, revisions to expected income look like

$$\mathbb{E}_t y_{t+j} - \mathbb{E}_{t-1} y_{t+j} = \begin{cases} \varepsilon_t & \text{if} \quad j = 0 \\ \lambda\varepsilon_t & \text{if} \quad j = 1 \\ 0 & \text{if} \quad j > 1 \end{cases}$$

Substituting this into (14) we get

$$\begin{aligned} \Delta c_t &= \frac{r}{1+r}\left([\mathbb{E}_t y_t - \mathbb{E}_{t-1} y_t] + \left(\frac{1}{1+r}\right)[\mathbb{E}_t y_{t+1} - \mathbb{E}_{t-1} y_{t+1}]\right) \\ &= \frac{r}{1+r}\left(1 + \frac{\lambda}{1+r}\right)\varepsilon_t \end{aligned}$$

From this example it is straight-forward to see what happens for any moving average representation:

$$y_t = \varepsilon_t + \sum_{j=1}^{\infty} \lambda_j \varepsilon_{t-j} = \varepsilon_t + \lambda_1 \varepsilon_{t-1} + \lambda_2 \varepsilon_{t-2} + \cdots$$

Then the warranted revision to consumption is

$$\Delta c_t = \frac{r}{1+r}\left(1 + \frac{\lambda_1}{1+r} + \frac{\lambda_2}{(1+r)^2} + \cdots\right)\varepsilon_t$$

This gives us a simple rule for evaluating consumption changes from the moving average representation of a time-series: simply discount the moving average terms, and add.

Suppose instead that $\{y_t\}_{t=0}^{\infty}$ is follows an AR(1) process:

$$y_t = \rho y_{t-1} + \varepsilon_t$$

with $\rho \in (-1, 1)$. Substituting this into (14) we get

$$\begin{aligned} \Delta c_t &= \frac{r}{1+r}\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j [\mathbb{E}_t y_{t+j} - \mathbb{E}_{t-1} y_{t+j}] \\ &= \frac{r}{1+r}\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j \rho^j \varepsilon_t \\ &= \frac{r}{1+r}\left(\frac{1}{1 - \frac{1}{1+r}\rho}\right)\varepsilon_t = \frac{r}{1+r}\left(\frac{1+r}{1+r-\rho}\right)\varepsilon_t \\ &= \frac{r}{1+r-\rho}\varepsilon_t \end{aligned}$$

Therefore a unit innovation in income causes permanent income to change by:

$$\frac{r}{1+r-\rho}$$

which is monotonically increasing in ρ. In fact, this approaches 1 as ρ approaches 1. This limit case is the case in which income itself follows a random walk. Then the best predictor of future income is current income, and innovations are expected to persist forever. In this case any innovation to income is permanent, and has a 1-for-1 effect on consumption.

These examples show how innovations in income translate into changes in permanent income, and thus into changes in consumption. The important point to take from these examples is that the effect of innovations in income on permanent income will be larger the more persistent the income process. Hence, the marginal propensity to consume from an innovation to current income depends crucially on the persistence of the income process. Temporary, or transitory, changes in income should produce only small effects on current consumption, whereas permanent or very persistent changes in income should result in large effects on current consumption.

How would one test this? Flavin (1981) tested equation (14), together with an autoregressive specification for the process governing income. She found that consumption responded to predictable changes in income—this is called "excess sensitivity," and hence violates the Permanent Income model.

Furthermore, Deaton (1987) showed that in the data the marginal propensity to consume out of very persistent income innovations appeared to be too small (given the persistence of the income process). We call this violation "excess smoothness." In the data it looks like aggregate consumption is too smooth relative to persistent changes in aggregate income.

Finally, in terms of studies on micro data as opposed to macro data, there are seminal papers by Hall and Mishkin (1982) and Zeldes (1989). See Attanasio and Weber (2010) for a fantastic survey of this literature, both theory and empirics.

5 Precautionary Savings: A Simple Two-Period Model

Let's now move away from the special case of quadratic utility.

We will now try to understand the household's optimal consumption and savings problem with an uncertain income stream, but now with more general utility functions. With more general utility functions we will break the certainty equivalence result that we get with quadratic utility.

To build intuition, let us first consider a very simple two-period savings problem

$$\max u(c_0) + \beta \mathbb{E}(c_1)$$

where

$$\begin{aligned} c_0 + a &= x \\ c_1 &= (1+r)a + y_1 \end{aligned}$$

with $u' > 0$, $u'' < 0$ and where endowment x is given by income y_1 is uncertain at time 0. Taking

FOCs, the Euler equation is obviously given by

$$u'(c_0) = \beta(1+r)\mathbb{E}u'(c_1)$$

or

$$u'(x_0 - a) = \beta(1+r)\mathbb{E}u'((1+r)a + y_1)$$

This along with the budget constraint would pin down the optimal choice of a^* for the agent. We can represent the optimal asset level graphically in Figure 1.

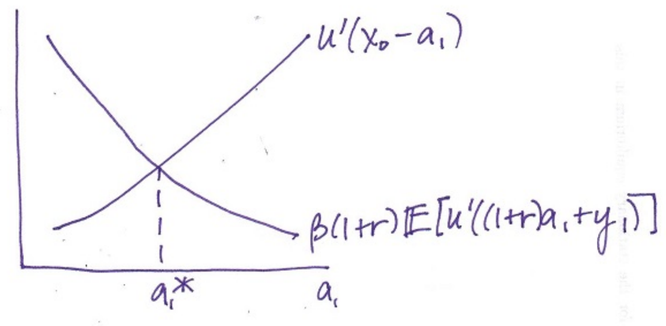

Figure 1. Optimal asset determination in the 2-period model

Now consider a mean-preserving spread ε to time 1 income. That is, replace y_1 with $\tilde{y}_1$ where

$$\tilde{y}_1 = y_1 + \varepsilon, \qquad \text{with} \qquad \mathbb{E}\varepsilon = 0$$

so that we keep the mean income constant but we just increase the variance (second moment).

There are three possibilities for what happens to the household's optimal asset level a:

1. if the function u' is linear, then a^* stays constant.
2. if u' is convex, then RHS of Euler equation rises and as a result a^* increases.
3. if u' is concave, then RHS of Euler equation falls and as a result a^* decreases.

These follow from Jensen's inequality. To see this, Figure 3 demonstrates Jensen's inequality for a convex marginal utility function u'.

From this we see that it is the third derivative of the utility function, u''', that matters for precautionary savings! While this is not the most intuitive utility feature to understand, some people call this derivative "prudence."

By introspection, people seem to increase their precautionary savings when variance of income goes up. So it seems natural that $u''' > 0$. Furthermore, note that for CRRA: $u'(c) = c^{-\gamma}$ with $\gamma > 0$, thus u' is convex.

For quadratic utility, marginal utility is linear. That's why in our analysis above of the permanent income hypothesis only the mean of the income process matters (not variance nor any other higher order moments).

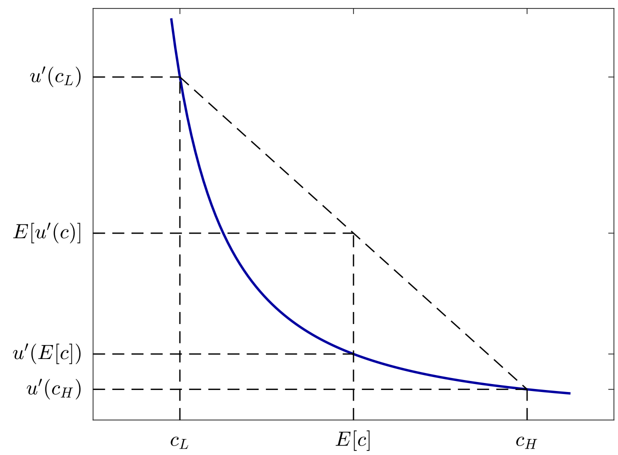

Figure 2. Jensen's Inequality with convex u'

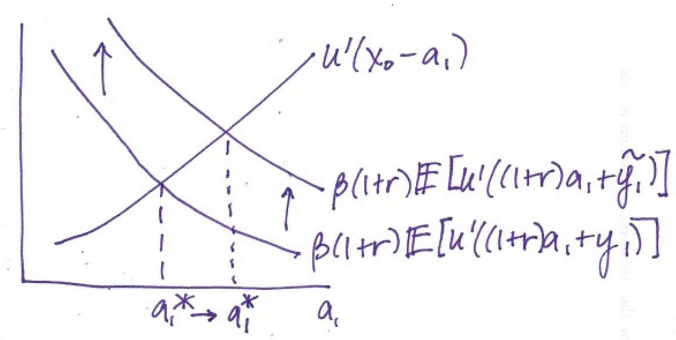

Figure 3. Optimal asset determination in the 2-period model with mean preserving spread

Finally, note that even if we consider a utility function that features a marginal utility that is not convex globally, it must ultimately be convex at the limits. Because if $u'(c) > 0, u''(c) < 0$ and $c \geq 0$, it must be the case that $u'(c)$ is strictly convex near the limits of $c \to 0$ and $c \to \infty$. Thus, convex marginal utility is somewhat unavoidable at least in the limits.

References

Ljungqvist, Lars and Thomas J. Sargent, *Recursive macroeconomic theory*, Cambridge, Mass: MIT Press, 2004.