Heterogeneous Agent Models

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We continue with our exploration of an incomplete markets model in which the household has a precautionary savings motive to self-insure against idiosyncratic income risk. This lecture builds on material found in Chapters 16 and 17 of Ljungqvist and Sargent (2004).

1 The Individual Household's Problem

We continue to maintain the assumption that $\beta(1+r) = 1$. Recall that the household faces a stochastic endowment process$\{y_t\}_{t=0}^{\infty}$ in which in each period, the endowment takes one of a finite number of values, indexed by $s \in S$:

$$y \in \mathcal{Y} = \{\bar{y}_1, \bar{y}_2, \ldots \bar{y}_S\}$$

where without loss of generality we assume

$$\bar{y}_1 < \bar{y}_2 < \ldots < \bar{y}_S.$$

We assume that the sequence of endowments are i.i.d. over time and drawn from the following distribution

$$\Pr(y_t = \bar{y}_s) = \pi_s$$

with $\sum \pi_s = 1$.

We let a_t denote the assets of the consumer at the beginning of in period t *including* the current realization of the income process. Then the household's Bellman equation can be written as:

$$V(a) = \max_{c \in [0,a]} \left\{ u(c) + \beta \sum_{s \in S} \pi_s V((1+r)(a-c) + \bar{y}_s) \right\}$$

recall that $c \leq a$ is the borrowing constraint. Recall that the value function $V(a)$ inherits the basic properties of $u(c)$; that is, it is increasing, strictly concave, and differentiable.

Household optimization. The FOC for the household's problem is given by:

$$u'(c) = \beta(1+r) \sum_{s \in S} \pi_s V'((1+r)(a-c) + \bar{y}_s) + \lambda,$$

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where λ is the Lagrange multiplier on the borrowing constraint: λ is strictly positive if the borrowing constraint is binding, or equal to zero if it is non-binding.

We may thus rewrite the Euler equation as follows

$$u'(c) \geq \beta(1+r) \sum_{s \in S} \pi_s V'((1+r)(a-c) + \bar{y}_s)$$

with equality if the no-borrowing constraint is not biding. Benveniste-Scheinkman gives us:

$$u'(c) = V'(a)$$

so that the Euler equation can be written as

$$V'(a) \geq \sum_{s \in S} \beta(1+r) \pi_s V'(a'_s)$$

where a'_s is next period's assets if the income shock is $\bar{y}_s$:

$$a'_s = (1+r)(a-c) + \bar{y}_s$$

Imposing that $\beta(1+r) = 1$, we have that $\{V'(a)\}$ is a nonnegative supermartingale:

$$\mathbb{E}[V'(a')] \leq V'(a)$$

The expected value of $V'(a')$ is weakly less than its value today. [If instead $\mathbb{E}[V'(a')] \geq V'(a)$, we would say that $\{V'(a)\}$ is a submartingale.]

We then use the following theorem from Doob (1953).

Theorem 1. *The Supermartingale Convergence Theorem (Doob, 1953). Let $\{Z_t\}$ be a nonnegative supermartingale. Then there exists a random variable Z such that*

$$\lim_{t \to \infty} Z_t = Z$$

almost surely and $E[Z] < \infty$ i.e. Z_t converges almost surely to a finite limit.

Proof. Omitted. □

In other words, if Z_t is a nonnegative supermartingale, it converges almost surely to a finite limit. Applying this to our problem, we reach the following result.

Proposition 1. $\{V'(a)\}$ *converges almost surely to a finite limit. The limit value of $V'(a)$ is zero, and hence in the limit assets diverge to infinity.*

Proof. First, $\{V'(a)\}$ converges almost surely to a finite limit by the supermartingale convergence theorem. Next, the limit value of $V'(a)$ must be zero because of the following argument.

Suppose that instead $V'(a)$ converges to a strictly positive limit:

$$\lim_{t \to \infty} V'(a) > 0$$

Let a^* be the associated optimal path for asset accumulation. Then

$$\lim_{t\to\infty} a_t^* = \bar{a}$$

where

$$0 < \bar{a} < \infty$$

Therefore a_t converges to a finite positive value. Let $c = g(a)$ be the optimal policy function for c. Then from the budget constraint we have

$$a_{t+1} = (1+r)(\bar{a} - g(\bar{a})) + y_{t+1}$$

but y_{t+1} is random. Therefore, a_{t+1} is not necessarily equal to $\bar{a}$. This is a contradiction, and hence $V'(a)$ cannot converge to a strcitly positive limit. Instead $V'(a)$ must converge to zero, implying that assets diverge to infinity. □

Note that assets do not necessarily increase monotonically. Low income realizations will reduce asset holdings. But over time, assets will grow to infinity. The fact that assets converge to infinity means that the individuals' consumption also converges to infinity.

To see this, note that from the Beneviste-Scheinkman formula we again have that:

$$u'(c) \geq \sum_{s\in S} \beta(1+r)\pi_s u'(c').$$

Imposing that $\beta(1+r) = 1$, we have that $\{u'(c)\}$ is a nonnegative supermartingale:

$$\mathbb{E}[u'(c')] \leq u'(c)$$

The expected value of $u'(c)$ is weakly less than its value today. Again by the supermartingale convergence theorem, $\{u'(c)\}$ converges almost surely to a finite limit.

Proposition 2. *$\{u'(c)\}$ converges almost surely to a finite limit. The limit value of $u'(c)$ is zero, and hence in the limit consumption diverges to infinity.*

Proof. Again suppose that $u'(c)$ converges to a strictly positive limit:

$$\lim_{t\to\infty} u'(c) > 0$$

Let c^* be the associated optimal path of consumption. Then

$$\lim_{t\to\infty} c_t^* = \bar{c} \in (0, \infty)$$

$\bar{c}$ would have to be the maximum constant consumption level sustainable for *all* possible future realizations of income.

But this would imply that $\bar{c}$ is the annuity value of current assets and a stream of future incomes all equal to the lowest possible realization, i.e. $y_{t+k} = \bar{y}_1$ for all $k > 0$. (why? due to the constraint that consumption cannot go negative).

However note that whenever y_{t+k} ends up being higher than the lowest possible realization, $\bar{y}_1$, the agent's wealth (i.e. current assets a) increases, implying that $\bar{c}$ will clearly not be the optimal consumption level anymore: the optimal level of ocnsumption would increase. It follows that consumption will vary with income realizations. But this is a contradiction: consumption cannot converge to a finite limit.

Therefore, it cannot be the case that consumption converges to a finite limit. $u'(c)$ must converge to zero, and consumption must diverge to infinity. □

1.1 Summary: Certainty vs. Uncertainty

Recall that under certainty, the optimal consumption path converges to a finite limit as long as the discounted value of all future income is bounded across t.

$$\lim_{t\to\infty} c_t^* = \bar{c} \in (0, \infty)$$

where

$$\bar{c} = \frac{r}{1+r} \sup_t x_t$$

where

$$x_t \equiv \sum_{j=t}^{\infty} \left(\frac{1}{1+r}\right)^{t-j} y_j$$

denotes permanent income. That is, under certainty the limiting value of the consumption path is given by the highest annuity value of the endowment process across all starting dates t.

In stark contrast, under uncertainty (a stochastic endowment process), we find that the optimal consumption path diverges to infinity!

$$\{c_t^*\} \to \infty$$

Assets also diverge to infinity. The stark difference between the two cases is remarkable!

Intuition. What is the intuition for this? Again the intuition for these results is not incredibly obvious. However it stems from what we discussed in the last class in the simple two period model. Due to the precautionary savings motive, agents save so much that both assets and consumption diverge to infinity. The optimality of unbounded consumption growth applies to utility functions whose marginal utility of consumption is strictly convex. However, even utility functions that do not have convex marginal utility globally must ultimately conform to a similar condition over long intervals in the limits (as we discussed in the previous class). See section 16.7 in Ljungqvist and Sargent (2004) for a discussion.

Note that throughout all of this analysis we have maintained the assumption that the interest rate is equal to the discount rate. Consider the Euler equation under certainty:

$$u'(c_t) = \beta(1+r)u'(c_{t+1})$$

With $\beta(1+r) = 1$ you get that $u'(c_t) = u'(c_{t+1})$ and hence $c_t = c_{t+1}$. However, once you add uncertainty and a borrowing constraint (tighter than the natural borrowing constraint), agents have a greater demand for assets due to the precautionary savings motive—they want to self-insure against bad shocks. Thus, agents demand a greater amount of assets, and if the interest rate does not fall in equilibrium to counteract this increase in demand, assets continue to rise and (almost surely) diverge to infinity.

2 The Interest Rate

Consider the Euler Equation

$$u'(c_t) \geq \beta(1+r)u'(c_{t+1})$$

and let m_t denote:

$$m_t \equiv \beta^t(1+r)^t u'(c_t)$$

Then Euler Equation can be written as:

$$\mathbb{E}_t m_{t+1} \leq m_t$$

Therefore m_t is a non-negative supermartingale. If we apply the supermartingale converge theorem,

$$m_t \rightarrow_{a.s.} \bar{m}$$

Next, consider two cases: $\beta(1+r) \geq 1$ and $\beta(1+r) < 1$.

Suppose $\beta(1+r) \geq 1$. By the supermartingale converge theorem,

$$V'(a_t) \rightarrow_{a.s} 0 \qquad \text{and} \qquad u'(c_t) \rightarrow_{a.s} 0$$

which implies assets and consumption diverge to infinity:

$$a_t \rightarrow_{a.s} \infty \qquad \text{and} \qquad c_t \rightarrow_{a.s} \infty.$$

Suppose instead $\beta(1+r) < 1$. Then in this case $u'(c_t)$ can remain strictly positive and vary randomly while

$$\beta^t(1+r)^t \rightarrow_{a.s.} 0.$$

In this case, the average level of assets and consumption need not diverge to infinity and can remain finite (but vary randomly).

3 Heterogeneous Agent Models

We will now consider a continuum of agents in the economy facing the savings problem we have considered above. These agents all face idiosyncratic income risk, but there are no insurance markets for them to share this risk. They can only borrow and save in a safe asset but face a borrowing constraint. Hence, we call this a form of incomplete markets. Furthermore, we will now treat the interest rate $1 + r$ as an equilibrium object. This class of models with many agents and incomplete markets was first introduced by Bewley (1977, 1980).

We often refer to this class of models as Bewley-Imrohoroglu-Huggett-Aiyagari Models. Within this class we will consider a particularly important and often used model of a production economy in which agents face uninsurable idiosyncratic income risk but may borrow and save in capital. This model was introduced by Aiyagari (1994).

3.1 The Aiyagari (1994) Model

Suppose the household's labor at time t, state s_t, evolves according to an m-state Markov chain with transition matrix $\pi(s'|s)$. That is, $s_t \in S = \{s_1, s_2, \ldots, s_m\}$ where S is the vector of the m employment states. Think of these as the household's labor endowment. Without loss of generality

$$s_1 < s_2 < \ldots < s_m$$

If the realization of the process at time t is s_t, then at time t the household receives labor income ws_t. Thus employment opportunities determine the labor income process.

The household's problem is given by

$$\max \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + x_t = \tilde{r} k_t + w s_t$$

and

$$k_{t+1} = (1 - \delta) k_t + x_t$$

and a borrowing constraint which we will consider shortly. Let $\tilde{r}$ be the rental rate on capital and w is a competitive wage rate, to be determined in equilibrium, but the household takes these as given. We may combine these two equations to get

$$c_t + k_{t+1} = (1 + \tilde{r} - \delta) k_t + w s_t$$

Thus we may rewrite this as

$$c_t + k_{t+1} = (1 + r) k_t + w s_t$$

where $r = \tilde{r} - \delta$.

Digression: the general savings problem. For a moment let us use an assets notation, so that

$$c_t + a_{t+1} = (1+r)a_t + ws_t$$

This is the same as our model above, but with $a_t = k_t$. We may thus write the Bellman equation for the household as

$$V(a,s) = \max_{a'} u(c) + \beta \sum \pi(s'|s)V(a',s')$$

subject to

$$c + a' = (1+r)a + ws.$$

and a borrowing constraint

$$a' \geq -\phi$$

where $\phi \geq 0$ is more restrictive than the natural borrowing constraint.

The natural borrowing constraint. Suppose the household were not to consume for all periods going forward. Then iterating the budget constraint forward and imposing $c_{t+j} \geq 0$ for all j, we obtain the natural borrowing constriant,

$$a_t = -\sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^{j+1} ws_{t+j}$$

Finally, using the fact that $\min s_{t+j} = s_1$, we have

$$a_t = -\frac{ws_1}{r}$$

Ad hoc borrowing constraint. To impose a more restrictive borrowing constraint, we set

$$a' \geq -\phi$$

where ϕ satisfies

$$\phi < \frac{ws_1}{r},$$

i.e. it is tighter than the natural borrowing constraint.

3.2 Equilibrium in the Aiyagari Model

The Individual household's problem. We constrain the holdings of the asset into a grid:

$$\mathcal{A} \equiv \{-\phi, a_1, a_2, \ldots a_n\}$$

The household's Bellman equation can then be written as follows:

$$V(a,s) = \max_{a' \in \mathcal{A}} u((1+r)a + ws - a') + \beta \sum \pi(s'|s)V(a',s'). \tag{1}$$

Note that the state is now two-dimensional (a, s) because s is not i.i.d.

From this problem we can solve for the value function $V(a,s)$ that satisfies this equation as well as an associated policy function for asset holdings

$$a' = g(a,s).$$

Thus, g maps the period's two-dimensional state (a,s) into an optimal choice of assets a' to carry into the next period.

Wealth-employment distributions. All households in this economy at time t will have some state vector (a_t, s_t). Define the unconditional distribution of (a_t, s_t) pairs as

$$\lambda_t(a,s) = \Pr(a_t = a, s_t = s)$$

The exogenous Markov chain $\pi(s'|s)$ and the optimal policy function $a' = g(a,s)$ together induce a law of motion for the distribution λ_t given by

$$\begin{aligned}
\lambda_{t+1}(a',s') &= \Pr(a_{t+1} = a', s_{t+1} = s') \\
&= \sum_{a_t}\sum_{s_t} \Pr(a_{t+1} = a'|a_t, s_t)\Pr(s_{t+1} = s'|s_t)\lambda_t(a_t, s_t) \\
&= \sum_{a}\sum_{s} I(a'|a,s)\pi(s'|s)\lambda_t(a,s)
\end{aligned}$$

where

$$I(a'|a,s) = \begin{cases} 1 & \text{if} \quad a' = g(a,s) \\ 0 & \text{if} \quad a' \neq g(a,s) \end{cases}$$

We can also express this as

$$\lambda_{t+1}(a',s') = \sum_{s\in S}\sum_{\{a\in\mathcal{A}: a' = g(a,s)\}} \pi(s'|s)\lambda_t(a,s)$$

A time-invariant distribution λ that solves this equation, i.e. $\lambda_{t+1} = \lambda_t = \lambda$ for all t, is called a stationary distribution.

Similarly, let ξ be the invariant distribution associated with the Markov chain $\pi(s'|s)$. That is

$$\xi_{t+1}(s) = \sum_{s} \pi(s'|s)\xi_t(s)$$

Thus, ξ is the time-invariant distribution that solves this equation, i.e. $\xi_{t+1} = \xi_t = \xi$ for all t.

Aggregate Capital and Labor We assume that there is an initial distribution fo assets across households of $\lambda(a,s)$ where λ is the stationary distribution. Suppose the household's optimal policy function is $a' = g(a,s)$ for given interest rate r. Then the aggregate level of savings in the economy satisfies

$$A(r) = \sum_{a\in\mathcal{A}}\sum_{s\in S} \lambda(a,s)g(a,s)$$

Furthermore, assuming we start from the invariant distribution ξ over employment states, we have that the aggregate level of employment is

$$L = \xi' S.$$

where recall that $S = \{s_1, s_2, \ldots, s_m\}$ is the vector of the m employment states.

Finally, we assume there is a competitive representative firm with production function

$$F(K, L) = K^{\alpha} L^{1-\alpha}$$

with $\alpha \in (0, 1)$. From the firm's FOCs, we have that the wage rate and rental rate, respectively, satisfy:

$$w = \frac{\partial F(K,L)}{\partial L} \qquad \text{and} \qquad \tilde{r} = \frac{\partial F(K,L)}{\partial K}$$

Recall that $r = \tilde{r} - \delta$.

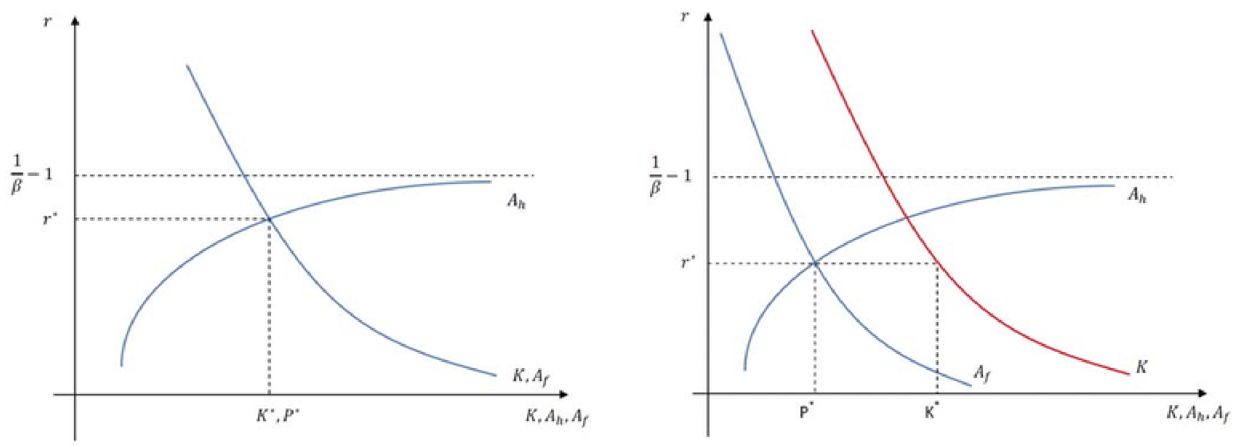

Figure 1. Equilibrium in Aiyagari (1994)

We reach the following definition of a stationary competitive equilibrium.

Definition 1. A stationary equilibrium is a policy function $a' = g(a, s)$, a probability distribution $\lambda(a, s)$, and aggregate capital, interest rate, and wage (K, r, w) such that

(i) the prices (r, w) satisfy

$$w = \frac{\partial F(K,L)}{\partial L} \qquad \text{and} \qquad r = \frac{\partial F(K,L)}{\partial K} - \delta$$

(ii) the policy function $g(a, s)$ solves the household's problem in 1 for given interest rate r;

(iii) the probability distribution $\lambda(a, s)$ is the stationary distribution associated with $(g(a, s), \pi)$;

(iv) the capital market clears (capital demand is equal to the supply of capital):

$$K = A(r)$$

where

$$A(r) = \sum_{a \in \mathcal{A}} \sum_{s \in S} \lambda(a, s) g(a, s)$$

The equilibrium is illustrated in the left panel of Figure 1.

References

Ljungqvist, Lars and Thomas J. Sargent, *Recursive macroeconomic theory*, Cambridge, Mass: MIT Press, 2004.

A Appendix

What is a martingale?

Definition 2. Suppose $\{\mathcal{F}_t\}$ is a filtration on some probability space $(\Omega, \mathcal{F}, P)$ where Ω is a sample space (the set of all possible outcomes), $\mathcal{F}$ is a set of events, and P is a probability measure (a function which maps these events into probabilities). Then, suppose that the sequence $\{Z_t\}$ satisfies the following properties:
(i) Z_t is measurable with respect to $\mathcal{F}_t$.
(ii) $\mathbb{E}|Z_t| < \infty$
(iii) $\mathbb{E}(Z_{t+1}|\mathcal{F}_t) = Z_t$

Then $\{Z_t\}$ is said to be a martingale with respect to the filtration $\mathcal{F}_t$. If instead $\mathbb{E}(Z_{t+1}|\mathcal{F}_t) \geq Z_t$ we say that $\{Z_t\}$ is a submartingale. If instead $\mathbb{E}(Z_{t+1}|\mathcal{F}_t) \leq Z_t$ we say that $\{Z_t\}$ is a supermartingale.