The Contraction Mapping Theorem

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In this lecture we state and prove the Contraction Mapping Theorem, also known as the Banach Fixed Point Theorem, a powerful fixed point theorem used often in economics. This theorem is regularly applied in dynamic settings in macro, industrial organization, structural labor, finance, and infinitely-repeated games. These lecture notes draw on material found in Chapter 3 of Stokey, Lucas and Prescott (1989) and Chapter C of Ok (2007).

1 What is a contraction?

We begin with the definition of a contraction.

Definition 1. Let (X, ρ) be a metric space and let $T : X \to X$ be a function mapping X into itself. We say T is a *contraction mapping with modulus* β if for some $\beta \in (0, 1) \subset \mathbb{R}$,

$$\rho(Tx, Ty) \leq \beta\rho(x, y), \qquad \forall x, y \in X.$$

Sometimes we call T a *self-map*: a function that maps from a set X back into X (the domain and codomain are identical). I will often refer to T as an *operator*.

Familiar examples of a contraction mapping are those on a closed interval on the real line. Let $X = [a, b] \subset \mathbb{R}$ with the Euclidean norm $\rho(x, y) = \|x - y\| = |x - y|$. Then $T : X \to X$ is a contraction if for some $\beta \in (0, 1)$,

$$|Tx - Ty| \leq \beta|x - y|, \qquad \forall x, y \in X.$$

For all $x \neq y$:

$$\frac{|Tx - Ty|}{|x - y|} \leq \beta.$$

That is, T is a contraction if it is a function with slope uniformly less than one in absolute value.

Let's consider an extremely simple example. Let $X = [0, 1] \subset \mathbb{R}$, with the Euclidean norm, $\rho(x, y) = |x - y|$. Consider the following operator $T : X \to X$, defined by:

$$Tx = .2 + .6x, \qquad \forall x \in [0, 1]. \tag{1}$$

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Then

$$|Tx - Ty| = .6|x - y|$$

Therefore

$$|Tx - Ty| \leq \beta \, |x - y|$$

for all $\beta \in [.6, 1)$. In this case we would say that T is a contraction mapping with modulus β, for any $\beta \in [.6, 1)$.

Next we would like to consider the fixed points of the contraction mapping T.

Definition 2. A fixed point of T is an element $x \in X$ that satisfies:

$$Tx = x.$$

In my example above (1), a fixed point satisfies:

$$.2 + .6x = x,$$

Solving this for x, we get $x = .5$. Furthermore, if you graph this simple function, you can easily see that this contraction has a unique fixed point.

2 The Contraction Mapping Theorem

Although the example above is super simple, the conclusion that the contraction has a unique fixed point is much more general. First, we define the iterations of applying T by

$$T^0 x = x, \qquad T^n x = T(T^{n-1}x), \qquad \forall n = 1, 2, \ldots$$

We now state and prove the Contraction Mapping Theorem, also known as the Banach Fixed Point Theorem.

Theorem 1. The Banach Fixed Point Theorem/The Contraction Mapping Theorem, general version.

Let (X, ρ) be a complete metric space. If $T : X \to X$ is a contraction mapping with modulus β, then:

(i) T has exactly one fixed point $x \in X$, and

(ii) for any $x_0 \in X$

$$\rho(T^n x_0, x) \leq \beta^n \rho(x_0, x), \qquad \forall n = 0, 1, 2, \ldots$$

Proof. Part (i). To prove the first part, we must find an $x \in X$ such that $Tx = x$ and show that no other element $x' \in X$ satisfies this criteria.

Choose an initial element $x_0 \in X$ and define the sequence $\{x_n\}_{n=0}^{\infty}$ by

$$x_n = T^n x_0$$

so that $x_{n+1} = Tx_n$, for all $n = 0, 1, 2, \ldots$.

The first step is to prove that the sequence we just defined is Cauchy. By the contraction property of T,

$$\rho(x_2, x_1) = \rho(Tx_1, Tx_0) \leq \beta\rho(x_1, x_0).$$

By induction, we get

$$\rho(x_{n+1}, x_n) \leq \beta^n \rho(x_1, x_0), \qquad \forall n = 1, 2, \ldots$$

Next, by the triangle inequality, for any $m > n$,

$$\rho(x_m, x_n) \leq \rho(x_m, x_{m-1}) + \cdots + \rho(x_{n+2}, x_{n+1}) + \rho(x_{n+1}, x_n)$$

Thus

$$\rho(x_m, x_n) \leq \left[\beta^{m-1} + \cdots + \beta^{n+1} + \beta^n\right] \rho(x_1, x_0) = \beta^n \left[\beta^{m-n-1} + \cdots + \beta + 1\right] \rho(x_1, x_0),$$

where

$$1 + \beta + \beta^2 \cdots + \beta^{m-n-1} < \frac{1}{1-\beta}.$$

Therefore

$$\rho(x_m, x_n) < \frac{\beta^n}{1-\beta}\rho(x_1, x_0) \tag{2}$$

It is clear from (2) that the sequence $\{x_n\}_{n=0}^{\infty}$ is Cauchy. Since (X, ρ) is a complete metric space, it follows that the sequence converges to an $x \in X$.

$$x_n \to x \in X.$$

Next we show that this limit satisfies $Tx = x$. Note that for all $n = 1, 2, \ldots$ and any $x_0 \in X$,the distance between Tx and x satisfies

$$\rho(Tx, x) \leq \rho(Tx, T^n x_0) + \rho(T^n x_0, x),$$

by the triangle inequality. Furthermore note that:

$$\rho(Tx, T^n x_0) = \rho(Tx, T(T^{n-1}x_0)) \leq \beta\rho(x, T^{n-1}x_0)$$

Therefore

$$\rho(Tx, x) \leq \beta\rho(x, T^{n-1}x_0) + \rho(T^n x_0, x).$$

But we have just shown that both terms on the right-hand side converge to zero as $n \to \infty$. Hence $\rho(Tx, x) = 0$, or equivalently, $Tx = x$. Therefore the limit of this sequence is a fixed point of T.

Next we must show that this fixed point is unique: that there is no other $x' \in X$ satisfying $Tx' = x'$. We can prove this by contradiction. Suppose there exists an $x' \in X$ satisfying $Tx' = x'$ and $x' \neq x$. Then

$$0 < a = \rho(x', x) = \rho(Tx', Tx) \leq \beta\rho(x', x) = \beta a$$

But $a = \beta a$ cannot hold for any $a > 0$ since $\beta < 1$. Therefore, the fixed point is unique.

Part (ii). Observe that first, for $n = 0$

$$\rho(T^0 x_0, x) = \rho(x_0, x).$$

For any $n = 1, 2, \ldots$,

$$\rho(T^n x_0, x) = \rho(T(T^{n-1} x_0), Tx) \leq \beta \rho(T^{n-1} x_0, x),$$

so that part (ii) of the theorem follows by induction. □

Part (i) of Theorem 1 states that a fixed point of a contraction on a complete metric space exists and is unique. Part (ii) of Theorem 1 states that starting from any intial point in $x_0 \in X$, repeated application of the contraction T gets you closer and closer to the unique fixed point.

Note that part (ii) of Theorem 1 bounds the distance between the nth iteration of the contraction and the fixed point, i.e. $\rho(T^n x_0, x)$, in terms of the distance between the initial initial element $x_0 \in X$ and the fixed point, i.e. $\rho(x_0, x)$.

However, this bound may not be very useful if the fixed point is not yet known, in which case you wouldn't know $\rho(x_0, x)$. The following result instead provides a computationally useful inequality.

Proposition 1. *Let (X, ρ) be a complete metric space, $T : X \to X$ a contraction mapping with modulus β,and $x \in X$ the unique fixed point of T. For any $x_0 \in X$,*

$$\rho(T^n x_0, x) \leq \frac{1}{1-\beta} \rho(T^n x_0, T^{n+1} x_0)$$

Exercise 1. Prove Proposition 1.

In contrast to part (ii) of Theorem 1, this proposition provides a bound that one can compute without knowledge of the fixed point x.

2.1 Fixed Points on function spaces.

In the example provided at the beginning of these lecture notes, we conceptualized the metric space (X, ρ) as a set in $\mathbb{R}$ with the Euclidean norm. However, in practice we most often apply the Contraction Mapping theorem to contractions in *function spaces.*

Towards this, I'm going to simply restate the theorem in the space of functions we are interested in. As in Lecture Notes 1, let $X \subseteq \mathbb{R}^N$. Let $C(X)$ be the set of bounded, continuous functions

$$f : X \to \mathbb{R}$$

with the sup norm

$$\|f\| = \sup_{x \in X} |f(x)|.$$

Furthermore, recall that at the end of Lecture Notes 1, we proved that $C(X)$ with the sup norm is a complete, normed vector space.

We next restate the Contraction Mapping Theorem in this particular context.

Theorem 2. The Banach Fixed Point Theorem/The Contraction Mapping Theorem, restated on $C(X)$.

Let $X \subseteq \mathbb{R}^N$ and let $C(X)$ be the space of bounded, continuous functions $f : X \to \mathbb{R}$ with the sup norm. Let $T : C(X) \to C(X)$ be a contraction mapping with modulus β; that is, for some $\beta \in (0,1) \subset \mathbb{R}$,

$$\|Tf - Tg\| \leq \beta \|f - g\|, \qquad \forall f, g \in C(X).$$

Then:

(i) T has exactly one fixed point $f \in C(X)$,

$$Tf = f, \text{ and}$$

(ii) for any $f_0 \in C(X)$,

$$\|T^n f_0 - f\| \leq \beta^n \|f_0 - f\|, \qquad \forall n = 0, 1, 2, \ldots$$

Proof. This follows from Theorem 1 and the fact that $C(X)$ with the sup norm is a complete normed vector space. □

3 Blackwell's Sufficient Conditions

In order to apply the contraction mapping theorem, one needs to not only check that the metric space is complete, but one must also verify that the particular operator T in question is a contraction. However, it is not always obvious whether a particular operator is or is not a contraction.

To solve this problem, one can often use the following sufficient conditions due to Blackwell.

Theorem 3. *(*Blackwell's sufficient conditions for a contraction.*) Let $X \subseteq \mathbb{R}^n$ and let $B(X)$ be the space of bounded functions $f : X \to \mathbb{R}$ with the sup norm. Let*

$$T : B(X) \to B(X)$$

be an operator satisfying the following two properties:

*(i) (*monotonicity*) $f, g \in B(X)$ and $f(x) \leq g(x)$, for all $x \in X$ implies*

$$Tf(x) \leq Tg(x), \qquad \forall x \in X;$$

*(ii) (*discounting*) there exists some $\beta \in (0,1)$ such that*

$$[T(f + a)](x) \leq Tf(x) + \beta a, \qquad \forall f \in B(X),\ a \in \mathbb{R}_+,\ x \in X,$$

where $f + a$ is the function defined by

$$(f + a)(x) = f(x) + a.$$

Then T is a contraction with modulus β.

Proof. Let $T : B(X) \to B(X)$ be an operator that satisfies the two conditions stated above. For any functions $f, g \in B(X)$,

$$f(x) - g(x) \leq \|f - g\|, \qquad \forall x \in X.$$

This implies

$$f(x) \leq g(x) + \|f - g\|, \qquad \forall x \in X.$$

Conditions (i) and (ii) imply

$$Tf(x) \leq T(g + \|f - g\|)(x) \leq Tg(x) + \beta \|f - g\|, \qquad \forall x \in X.$$

Therefore

$$Tf(x) - Tg(x) \leq \beta \|f - g\|, \qquad \forall x \in X.$$

Next, reversing the roles of f and g, by the same logic we get that

$$Tg(x) - Tf(x) \leq \beta \|f - g\|, \qquad \forall x \in X.$$

Together, these imply that

$$\|Tf - Tg\| = \sup_{x \in X} |Tf(x) - Tg(x)| \leq \beta \|f - g\|$$

as was to be shown. □

Blackwell's result will play a key role in allowing us to apply the Contraction Mapping Theorem to certain settings.

References

Ok, Efe A., "Real Analysis with Economic Applications," 2007.

Stokey, Nancy L., Robert E. Lucas, and Edward C. Prescott, *Recursive Methods in Economic Dynamics*, Harvard University Press, 1989.