The Real Business Cycle Model

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The Neoclassical Growth Model (NGM), a.k.a. the Ramsey-Cass-Koopmans model, was designed to depict how an economy might grow over time in a way that is consistent with growth facts. However, another huge innovation of the NGM is that it incorporates explicit microfoundations. That is, the NGM is a competitive general equilibrium model with utility-maximizing households, profit-maximizing firms, and prices that clear markets. It incorporates features that should be part of *any* plausible model, including:

- households like to consume and moreover like to smooth their consumption
- capital and labor are important inputs in production
- firms maximize profits, households maximize utility subject to budget constraints

The innovation that macro should be modeled as the general equilibrium of a micro-founded economy with individual households and firms means that we can represent the paths of aggregate consumption, investment, capital, output, and hours as market-clearing equilibrium outcomes (as we saw in the decentralized competitive equilibrium of the NGM).

We now move to business cycles. In the previous lecture we documented a number of prominent business cycle stylized facts. We now seek to develop a theoretical model that can help us understand and rationalize these facts. Towards this goal, we follow the direction of Kydland and Prescott (1982). We start with the Neoclassical Growth Model and modify it in two ways: (i) we allow for endogenous labor supply (or a labor-leisure choice), and (ii) we add a particular type of exogenous stochastic disturbance to technology.

1 The RBC Environment: Primitives

Here I set up the primitives of the Real Business Cycle economy, that is: technology, preferences, and resource constraints. This treatment is similar to that in Cooley and Prescott (1995).

Time is discrete and infinite: $t = 0, 1, \ldots$.

1.1 Production with Technology (TFP) Shocks

To formalize business cycles as a stochastic phenomenon, we need to introduce some source of uncertainty (some type of exogenous stochastic disturbance) into the model. But what kind of

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uncertainty? If you look at the planner's problem for the NGM, the only exogenous features in that problem are preferences and technologies. Hence, a starting point would be to introduce uncertainty into one of these features.

In the data, we saw that TFP as measured by by the Solow residual, fluctuates a lot over the business cycle. This motivates us to introduce TFP shocks into the NGM and study the behavior of the economy in response to these shocks. The goal is to see whether the observed TFP fluctuations (which may be endogenous in reality but we will take them as exogenous in the model) can generate fluctuations in all other macro variables (output, employment, consumption, etc) that are similar to the observed business cycle fluctuations.

We assume the firm's technology is now given by

$$Y_t = z_t F(K_t, L_t)$$

where z_t is Total Factor Productivity (TFP) and F continues to be a neoclassical production function satisfying all of the regularity conditions previously stated.

We formalize uncertainty over TFP in the following way. In each period $t = 0, 1, \ldots$, the economy experiences one of finitely many events s_t. We let:

$$s_t \in S$$

where S is a finite set. We write realized productivity at time t as a function of the current state: $z_t = z(s_t)$, where:

$$z : S \to \mathbb{R}_+.$$

Because of the introduction of uncertainty, all economic variables in any given date may depend not only on the calendar date t, and the current state s_t, but may also depend on the entire *history* of events up to this date. We denote by

$$s^t = (s_0, s_1, ..., s_{t-1}, s_t) \in S^t$$

the history of events up to and including period t, where

$$S^t \equiv S \times S \times \cdots \times S.$$

We let all economic variables in the model be contingent on this history; equivalently, you can think of the paths of consumption, investment etc as stochastic processes (stochastic paths). In other words, there is a natural commodity space in which all goods are differentiated by histories.

We let $\pi(s^t|s^{t-1})$ denote the probability of history s^t conditional on s^{t-1}. With some abuse of notation, we denote the ex-ante unconditional probability of history s^t by $\pi(s^t)$. We assume the initial realization s_0 is given, i.e. $\pi(s_0) = 1$.

1.2 Preferences

We further modify preferences so that households now value leisure and face a labor-leisure trade-off. The reason for adding this feature is to see if we can generate movements in labor over the business cycle.

There is a continuum of infinitely-lived households of unit mass, indexed by $i \in I \equiv [0,1]$. Households preferences are identical and given by their expected utility over all paths of consumption and labor.

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t U(c_t^i, \ell_t^i) = \sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t \pi(s^t) U(c_t^i(s^t), \ell_t^i(s^t)) \tag{1}$$

where $c_t^i(s^t)$ and $\ell_t^i(s^t)$ is the consumption and labor of household i at time t, history s^t. We assume that the flow (per-period) utility function U is continuously differentiable in both arguments, it is increasing and concave in the first argument but decreasing and concave in the second argument:

$$U_\ell < 0 \qquad \text{and} \qquad U_{\ell\ell} < 0$$

along with the Inada conditions (in order to ensure an interior solution).

1.3 Resource Contraints

We now let capital letters denote aggregate quantities and lower case letters denote individual household quantities:

$$C_t(s^t) = \int c_t^i(s^t)di, \quad L_t(s^t) = \int \ell_t^i(s^t)di, \quad K_t(s^t) = \int k_t^i(s^t)di, \qquad \forall t,$$

where k_t^i is the physical capital holdings of household i at time t. The aggregate resource constraint is given by

$$C_t(s^t) + K_{t+1}(s^t) = z(s_t)F(K_t(s^{t-1}), L_t(s^t)) + (1-\delta)K_t(s^{t-1}), \qquad \forall t.$$

2 The Planner's Sequence Problem

We consider the planner's problem for this economy. In particular, we consider a planner choosing a complete contingent plan of individual household consumption, labor, and capital, as well as the aggregate allocation of these objects:

$$\left\{(c_t^i(s^t), \ell_t^i(s^t), k_{t+1}^i(s^t))_{i\in I}; C_t(s^t), L_t(s^t), K_{t+1}(s^t)\right\}_{s^t\in S^t}$$

However, to make our lives simpler, note that the planner will necessarily choose:

$$c_t^i(s^t) = C_t(s^t), \quad \ell_t^i(s^t) = L_t(s^t), \quad k_{t+1}^i(s^t) = K_{t+1}(s^t), \qquad \forall s^t \in S^t.$$

That is, all households are ex-ante identical so the planner would optimally choose the same allocation for each household. We will use this observation to simplify our problem—we henceforth drop the i superscript.

There are two formulations for this problem; we first consider the sequence problem and afteward discuss the recursive formulation.We write the planner's sequence problem as follows.

Planner's Sequence Problem. Given initial $k_0 > 0$, the planner chooses a complete contingent plan

$$\left\{c_t(s^t), \ell_t(s^t), k_{t+1}(s^t)\right\}_{s^t \in S^t}$$

so as to maximize

$$\max \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) U(c_t(s^t), \ell_t(s^t))$$

subject to the resource constraint

$$c_t(s^t) + k_{t+1}(s^t) = z(s_t) F(k_t(s^{t-1}), \ell_t(s^t)) + (1-\delta) k_t(s^{t-1}), \qquad \forall s^t \in S^t, \tag{2}$$

and non-negativity constraints on consumption and capital:

$$c_t(s^t) \geq 0, \qquad k_{t+1}(s^t) \geq 0, \qquad \forall s^t \in S^t. \tag{3}$$

We can think of the planner in period 0 as choosing an infinite sequence that describes all future consumption, labor, and capital allocations *for all* histories s^t. That is, the planner chooses a complete, *contingent plan* for the allocation.

Now that we have added shocks, note that the resource constraint (2) must hold not only at every date t but also in *every possible history*, s^t.

For each history s^t, let $\beta^t \pi(s^t) \lambda(s^t)$ be the multiplier attached to this constraint. We write the Lagrangian for the planner's problem as follows.

$$\begin{aligned} \mathcal{L} \;\; = \;\; & \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) U(c_t(s^t), \ell_t(s^t)) \\ & - \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) \lambda(s^t) \left[c_t(s^t) + k_{t+1}(s^t) - z(s_t) F(k_t(s^{t-1}), \ell_t(s^t)) - (1-\delta) k_t(s^{t-1})\right] \end{aligned}$$

In what follows, I will use the following shorthand notation:

$$U_c(s^t) = \left.\frac{\partial U(c, \ell)}{\partial c}\right|_{c=c_t(s^t), \ell=\ell_t(s^t)}$$

$$U_\ell(s^t) = \left.\frac{\partial U(c, \ell)}{\partial \ell}\right|_{c=c_t(s^t), \ell=\ell_t(s^t)}$$

for the marginal utility of consumption and the marginal (dis)utility of labor, respectively. Similarly:

$$F_L(s^t) = \left.\frac{\partial F(K,L)}{\partial L}\right|_{K=k_t(s^{t-1}),L=\ell_t(s^t)}$$
$$F_K(s^t) = \left.\frac{\partial F(K,L)}{\partial K}\right|_{K=k_t(s^{t-1}),L=\ell_t(s^t)}$$

for the marginal product of labor and the marginal product of capital, respectively.

The first-order conditions of the planner's problem are given by

$$\begin{aligned}
c_t &: \quad \beta^t \pi(s^t) U_c(s^t) - \beta^t \pi(s^t)\lambda(s^t) = 0 \\
\ell_t &: \quad \beta^t \pi(s^t) U_\ell(s^t) + \beta^t \pi(s^t)\lambda(s^t) z(s_t) F_L(s^t) = 0 \\
k_{t+1} &: \quad -\beta^t \pi(s^t)\lambda_t(s^t) + \beta^{t+1} \sum_{s^{t+1}|s^t} \pi(s^{t+1})\lambda(s^{t+1}) \left[1 + z(s_{t+1}) F_K(s^{t+1}) - \delta\right] = 0
\end{aligned}$$

Combining these FOCs yield the following conditions.

The Lagrange Multiplier. The multiplier is equal to the marginal utility of consumption:

$$\lambda(s^t) = U_c(s^t), \qquad \forall s^t \in S^t.$$

That is, the planner's shadow value of resources at time t, history s^t is given by the marginal utility of consumption.

The Intertemporal Condition. Also known as the Euler Equation. The optimality condition between consumption today and consumption tomorrow is given by the following:

$$U_c(s^t) = \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left[U_c(s^{t+1}) \left(1 + z(s_{t+1}) F_K(s^{t+1}) - \delta\right)\right], \qquad \forall s^t \in S^t$$

where I have used Bayes rule

$$\pi(s^{t+1}|s^t) = \frac{\pi(s^{t+1})}{\pi(s^t)}.$$

We may rewrite this as follows

$$U_c(s^t) = \beta \mathbb{E}\left[U_c(s^{t+1}) \left(1 + z(s_{t+1}) F_K(s^{t+1}) - \delta\right) |s^t\right], \qquad \forall s^t \in S^t.$$

Giving up one unit of consumption today results in an extra unit of capital for tomorrow. Thus, the marginal utility of consumption today must equal the expected marginal benefit (value) of this extra unit of capital. The expected marginal value of an extra unit of capital tomorrow is the expected marginal return on capital in terms of output times the marginal utility of consumption, discounted by the discount factor. It is just another way of saying that the marginal cost today of one unit less of consumption must equal the expected marginal benefit.

The Intratemporal Condition. The optimality condition between consumption and labor is given by:

$$-U_\ell(s^t) = U_c(s^t) z(s_t) F_L(s^t), \qquad \forall s^t \in S^t.$$

The marginal disutility of labor must equal the marginal utility of consumption multiplied by the marginal product of labor, state-by-state. That is, the marginal cost of labor is equal to the marginal benefit. Another way to write this would be to divide through by $U_c(s^t)$ as follows:

$$-\frac{U_\ell(s^t)}{U_c(s^t)} = z(s_t) F_L(s^t), \qquad \forall s^t \in S^t.$$

The left hand side is the marginal rate of substitution between labor and consumption, and the right hand side is simply the marginal rate of transformation. The marginal rate of transformation of labor into consumption is the marginal product of labor, i.e. $z(s_t)F_L(s^t)$. Optimality equates the MRS with the MRT in every state.

The Transversality condition. As in the NGM, the planner's transversality condition is given by:

$$\lim_{t\to\infty} \sum_{s^t} \beta^t \pi(s^t) U_c(s^t) k_{t+1}(s^t) = 0,$$

or alternatively:

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[U_c(s^t) k_{t+1}(s^t)\right] = 0.$$

Solution to the Planner's problem. The planner's optimum is thereby given by the unique contingent plan that satisfies the above optimality conditions along with the economy's resource constraint, the initial condition, and an expected value version of the transversality condition (essentially the same as in the NGM deterministic case).

Proposition 1. *The socially optimal allocation*

$$\left\{c_t(s^t), \ell_t(s^t), k_{t+1}(s^t)\right\}_{s^t \in S^t}$$

is the unique contingent plan that satisfies, for every history, the following conditions:

$$U_c(s^t) = \beta \mathbb{E}\left[U_c(s^{t+1})(1 + z(s_{t+1})F_K(s^{t+1}) - \delta) | s^t\right], \qquad \forall s^t \in S^t; \tag{4}$$

$$-\frac{U_\ell(s^t)}{U_c(s^t)} = z(s_t) F_L(s^t), \qquad \forall s^t \in S^t; \tag{5}$$

and

$$c_t(s^t) + k_{t+1}(s^t) = z(s_t) F(k_t(s^{t-1}), \ell_t(s^t)) + (1-\delta) k_t(s^{t-1}), \qquad \forall s^t \in S^t; \tag{6}$$

along with the following boundary conditions:

$$k_0 > 0 \quad \textit{given}, \quad \textit{and} \quad \lim_{t\to\infty} \beta^t \mathbb{E}\left[U_c(s^t) k_{t+1}(s^t)\right] = 0. \tag{7}$$

The solution to the planner's problem in the RBC has similarities with the solution to the planner's problem in the NGM. Condition (6) is the resource constraint, summarizing feasibility of the allocation (aside from the non-negativity constraints). It now must hold not only at every date, but in every history s^t.

Condition (4) is the Euler equation, or the intertemporal optimality condition. it says that the marginal cost of consuming one less unit of consumption today must equal the expected marginal benefit. The marginal cost is marginal utility of consumption today. The marginal benefit is the marginal value of an extra unit of capital tomorrow—that is, the marginal return on capital in terms of output times the marginal utility of consumption, discounted by the discount factor. The only difference between this condition and what we had in the NGM is the conditional expectation operator, $\mathbb{E}\left[\cdot|s^t\right]$.

Condition (5) is the only truly "new" equation. We call it the intratemporal optimality condition. It states that the marginal rate of substitution between labor and consumption is equal to the marginal rate of transformation.

3 The Recursive Formulation of the Planner's Problem

We look for a stationary value function. For this we need to make a stationarity assumption for the stochastic process for TFP. Specifically, we do not want the statistical properties of z to change over time.

We assume that z_t follows a Markov process with transition (conditional) probabilities $\pi(z_{t+1}|z_t)$.

Definition 1. A Markov process is a stochastic process describing a sequence of possible events in which the probability of each event depends only on the state attained in the previous event.

$$\pi(z_{t+1}|z_t, z_{t-1}, z_{t-2}...) = \pi(z_{t+1}|z_t)$$

- Example 1: z_t fluctuates between two states, a high state and a low state. That is, $z = h$ (high) and $z = \ell$ (low) with a transition matrix

$$\begin{bmatrix} \pi_{hh} & \pi_{h\ell} \\ \pi_{\ell h} & \pi_{\ell\ell} \end{bmatrix}$$

- Example 2: the log of z follows an exogenous AR(1) process

$$\log z_{t+1} = \rho \log z_t + \varepsilon_{t+1} \qquad \varepsilon_{t+1} \sim \mathcal{N}(0, \sigma^2)$$

  with $\rho \in (0, 1)$.

If the productivity shock is Markov, that implies that a sufficient statistic for tomorrow's productivity shock is today's.

Corresponding to the sequence problem presented above, we may thereby write the planner's problem recursively in terms of the following Bellman equation.

Planner's Bellman Equation.

$$\begin{aligned} V(k,z) &= \max_{c,\ell,k'} U(c,\ell) + \beta \sum_{z'|z} \pi(z'|z) V(k',z') \\ s.t. \quad c + k' &= (1-\delta)k + zF(k,\ell) \end{aligned}$$

Here the state variables are k, z. The control variables are c, ℓ, k'. As we did before, let us rewrite the Bellman equation by reducing it to only two control variables, ℓ, k' as follows

$$V(k,z) = \max_{\ell,k' \in \Gamma(k,z)} U\left((1-\delta)k + zF(k,\ell) - k', \ell\right) + \beta \sum_{z'|z} \pi(z'|z) V(k',z') \tag{8}$$

What is the feasible set for (ℓ, k')? The feasible set is as follows

$$(\ell, k') \in \Gamma(k,z)$$

where

$$\Gamma(k,z) \equiv \left\{(\ell,k') | \, \ell \in \left[0, \overline{\ell}\right], \, k' \in [0, (1-\delta)k + zF(k,\ell)]\right\}$$

where we have assumed an upper bound on labor of $\overline{\ell} > 0$. One can make sense of this upper bound as there are only 24 hours in a day and hence only so many possible worker-hours per quarter. (Sufficient convexity of the disutility of labor should make it so that we never hit this upper bound anyway). We often normalize this upper bound to $\overline{\ell} = 1$.

Note that due to the stationarity assumption, the value function V does not change over time. The state variables k and z change, but the value function itself is fixed.

Using the same techniques as before, the FOCs to the planner's problem in (10) with respect to k' and ℓ are given by, respectively,

$$- U_c(c,\ell) + \beta \sum_{z'|z} \pi(z'|z) V_k(k',z') = 0 \tag{9}$$

and

$$U_\ell(c,\ell) + U_c(c,\ell) z F_L(k,\ell) = 0 \tag{10}$$

along with the Benveniste-Scheinkman (envelope) condition,

$$V_k(k,z) = (1 - \delta + zF_K(k,\ell)) U_c(c,\ell). \tag{11}$$

We thus derive following optimality conditions for the planner. First, from (10), the intratemporal condition given by

$$-\frac{U_\ell(c,\ell)}{U_c(c,\ell)} = zF_L(k,\ell).$$

Second, by combining (9) with the Benveniste-Scheinkman condition in (11), we get the intertemporal condition, a.k.a. the Euler equation

$$U_c(c,\ell) = \beta \sum_{z'|z} \pi(z'|z)(1-\delta+z'F_K(k',\ell'))U_c(c',\ell')$$

As we have shown before, the two formulations of the problem—the sequence problem and the recursive formulation—are equivalent and deliver the same solution under some technical restrictions. (Essentially you need to extend the earlier proofs to measurable spaces and measurable functions. See Chapters 7-9 of Stokey, Lucas and Prescott (1989) for details.)

4 Decentralization

Thus far we have focused on the characterization of the planner's problem—that is, on a fictitious scenario in which a benevolent social planner directly controls allocations. We now turn attention to the decentralized competitive equilibrium in this environment.

4.1 The Decentralized Environment

Similar to the decentralization of the NGM, in this economy there are households and firms. There is a competitive labor market in which households supply their labor to firms and earn a real wage rate denoted by w_t (expressed in terms of period-t consumption).

Capital is owned directly by the households. There is a competitive capital market in which households rent capital to the firms and earn a per-period real rental rate denoted by r_t. Both the market for labor and the market for capital are perfectly competitive; that is, all households and all firms are price takers in these markets.

Firms employ labor and rent capital in competitive labor and capital markets, have access to the same technology, and produce a homogeneous good that they sell competitively to the households.

Finally, households can trade a riskless, one-period bond, with one another. The bond is in zero net supply: households may borrow and lend from one another at an interest rate which is henceforth denoted by R_t.

Firms. There is a continuum of identical firms which we aggregate into a representative firm. The representative firm's technology is given by

$$Y_t = z_t F(K_t, L_t)$$

where z_t represents an exogenous shock to TFP and F continues to be a neoclassical production function that satisfies all regularity conditions.

In each period, the representative firm employs labor, rents capital, and produces a homogeneous good that is sold in a competitive market to the households. We therefore treat the firm's problem as a period-by-period and state-by-state profit maximization problem given by:

$$\max_{K_t, L_t} z_t F(K_t, L_t) - r_t K_t - w_t L_t.$$

Households. There is a continuum of households of unit mass, indexed by $i \in [0,1]$. Household preferences are identical and given by (1). Each household is endowed with initial capital level k_0. Let k_t^i denote the capital stock owned by household i at the beginning of period t and let b_t^i be its corresponding position in the bond market.

The household takes its initial capital level $k_0 > 0$ and initial bond position $b_0 = 0$ as given. We express the household's problem as follows.

Household's Problem. The household chooses a complete contingent plan,

$$\left\{c_t(s^t), \ell_t(s^t), k_{t+1}(s^t), b_{t+1}(s^t)\right\}_{s^t \in S^t}$$

in order to maximize lifetime expected utility

$$\max \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) U(c_t(s^t), \ell_t(s^t))$$

subject to:

$$c_t(s^t)+k_{t+1}(s^t)+b_{t+1}(s^t) = r_t(s^t)k_t(s^{t-1})+w_t(s^t)\ell_t(s^t)+(1-\delta)k_t(s^{t-1})+(1+R_t(s^{t-1}))b_t(s^{t-1}), \quad \forall t, s^t \tag{12}$$

$$c_t(s^t) \geq 0, \qquad k_{t+1}(s^t) \geq 0, \qquad \forall t, s^t$$

$$b_{t+1}(s^t) \geq \underline{b}, \qquad \forall t, s^t.$$

with

$$k_0 > 0 \quad \text{and} \quad b_0 = 0 \quad \text{given.}$$

The period-t budget constraint of the household is given by (12). That is, the household uses its labor income, capital income, and savings to finance consumption or investment in either physical capital or the risk-free bond.

Here I make the assumption that the return on bonds are non-state-contingent. Thus, $R_{t+1}(s^t)$ is the riskless rate; for this reason it is measurable in s^t.

Note furthermore that for the hosuehold's problem, I have added an "ad hoc" borrowing constraint of the form

$$b_{t+1}(s^t) \geq \underline{b}, \qquad \forall t, s^t, \tag{13}$$

where $\underline{b} < 0$ is an arbitrary limit on how much the household can borrow in every period.

We say that such a sequence, or plan, $\{c_t^i(s^t), \ell_t^i(s^t), k_{t+1}^i(s^t), b_{t+1}^i(s^t)\}_{t=0}^{T}$ is optimal for the household if it solves the household's problem.

Market Clearing. We assume that labor, capital, bond, and goods market must clear every period:

$$C_t(s^t) = \int c_t^i(s^t)di, \quad L_t(s^t) = \int \ell_t^i(s^t)di, \quad K_{t+1}(s^t) = \int k_{t+1}^i(s^t)di,$$

and bonds are in zero net supply:

$$\int b_{it}(s^t)di = 0$$

The representative household and the representative firm. Because all households are identical, we will consider the "representative household" of the economy. We will also consider the "representative firm." This implies:

$$c_t^i(s^t) = C_t(s^t), \quad \ell_t^i(s^t) = L_t(s^t), \quad k_{t+1}^i(s^t) = K_{t+1}(s^t), \qquad \forall s^t \in S^t. \tag{14}$$

and we henceforth drop the i superscript.

4.2 Equilibrium Definition

We define a competitive equilibrium in this economy as follows.

Definition 2. An equilibrium is a state-contingent price sequence

$$\{w_t(s^t), r_t(s^t), R_{t+1}(s^t)\}_{s^t \in S^t}$$

a state-contingent plan for the representative household,

$$\{c_t(s^t), \ell_t(s^t), k_{t+1}(s^t), b_{t+1}(s^t)\}_{s^t \in S^t},$$

and a state-contingent plan for the representative firm,

$$\{K_t(s^{t-1}), L_t(s^t)\}_{s^t \in S^t},$$

such that the following hold:

(i) given the price sequence, the sequence $\{c_t(s^t), \ell_t(s^t), k_{t+1}(s^t), b_{t+1}(s^t)\}$ solves the representative household's problem,

(ii) given the price sequence, the sequence $\{K_t(s^{t-1}), L_t(s^t)\}$ solves the representative firm's problem, and

(iii) markets clear.

Remarks. Note that the equilibrium is a fixed point between prices and allocations. This distinguishes it from the planner's solution, which was defined as a feasible allocation that maximizes welfare. Also note that unlike the definition of the planner's problem, the definition of an equilibrium does not a priori *require* that the allocation be feasible. However, any allocation that clears all markets trivially satisfies the economy's resource constraints, which means that any equilibrium allocation is indeed feasible.

Finally note that the solution concept requires that in any given moment the households know the prices that will clear the markets in all future states (firms just need to know the prices this period). This notion is what is known as "Rational Expectations." The key assumption is that the subjective beliefs that any given agent forms about future prices (or any other endogenous economic outcome) coincide with the true, objective processes that these objects follow in equilibrium (as the result of the joint behavior of all agents).

4.3 Equilibrium Characterization

We now characterize a competitive equilibrium in this economy.

Firms. Consider first the firms. The firm's problem is relatively straightforward as it simply solves it period-by-period. This yields the following FOCs:

$$r_t(s^t) = z(s^t)F_K(s^t), \qquad \forall t, s^t \tag{15}$$

$$w_t(s^t) = z(s^t)F_L(s^t), \qquad \forall t, s^t \tag{16}$$

Therefore, the real wage and real rental rate must be equal to the marginal product of labor and the marginal product of capital, respectively.

Households. Next, consider the household's problem. The households in this economy face a much more difficult problem as they must take into account the entire sequence of prices in the future.

The household's problem yields the following optimality conditions. First we have that the household's intratemporal optimality condition is given by

$$-U_\ell(s^t) = w_t(s^t)U_c(s^t), \qquad \forall t, s^t$$

which we may rewrite as

$$-\frac{U_\ell(s^t)}{U_c(s^t)} = w_t(s^t), \qquad \forall t, s^t.$$

That is, the household finds it optimal to equate its marginal rate of substitution between labor and consumption with the real wage.

The household's intertemporal condition with respect to capital is given by

$$U_c(s^t) = \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left[(1 + r_{t+1}(s^{t+1}) - \delta)U_c(s^{t+1})\right], \qquad \forall t, s^t$$

and $k_{t+1}(s^t) \geq 0$ with complementary slackness. We can rewrite this as

$$U_c(s^t) = \beta\mathbb{E}_t \left[(1 + r_{t+1}(s^{t+1}) - \delta)U_c(s^{t+1}) \middle| \, s^t\right], \qquad \forall t, s^t$$

The household's intertemporal condition with respect to bonds is given by

$$U_c(s^t) = \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \left[(1 + R_{t+1}(s^t))U_c(s^{t+1})\right], \qquad \forall t, s^t$$

and $b_{t+1}(s^t) \geq \underline{b}$ with complementary slackness. Given that we have assumed bonds are riskless, this last condition may be rewritten as follows

$$U_c(s^t) = \beta(1 + R_{t+1}(s^t))\mathbb{E}_t \left[U_c(s^{t+1}) \middle| s^t \right], \qquad \forall t, s^t.$$

Finally the household's boundary conditions are given by the initial condition $k_0 > 0$ and the transversality condition:

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[U_c(s^t)k_{t+1}(s^t)\right] = 0. \tag{17}$$

From the budget constraint to the resource constraint. By letting $b_{t+1}(s^t) = 0$, we can henceforth reduce the budget constraint of the household to the following:

$$c_t(s^t) + k_{t+1}(s^t) = r_t(s^t)k_t(s^{t-1}) + w_t(s^t)\ell_t(s^t) + (1 - \delta)k_t(s^{t-1}), \qquad \forall s^t \in S^t.$$

Finally, if we replace the rental and wage rate from the firm's optimality conditions, (15) and (16), into the above equation we reach the following condition:

$$c_t(s^t) + k_{t+1}(s^t) = z(s^t)F_K(s^t)k_t(s^{t-1}) + z(s^t)F_L(s^t)\ell_t(s^t) + (1 - \delta)k_t(s^{t-1}), \qquad \forall s^t \in S^t.$$

Therefore

$$c_t(s^t) + k_{t+1}(s^t) = z(s_t)F(k_t(s^{t-1}), \ell_t(s^t)) + (1 - \delta)k_t(s^{t-1}), \qquad \forall s^t \in S^t;$$

which is the resource constraint of the economy as a whole.

Combining these findings, we obtain the following characterization of the equilibrium.

Proposition 2. *An allocation*

$$\left\{c_t(s^t), \ell_t(s^t), k_{t+1}(s^t)\right\}_{s^t \in S^t}$$

is part of an equilibrium if and only if it satisfies:

$$U_c(s^t) = \beta\mathbb{E}\left[U_c(s^{t+1})\left(1 + z(s_{t+1})F_K(s^{t+1}) - \delta\right)|s^t\right], \qquad \forall s^t \in S^t; \tag{18}$$

$$-\frac{U_\ell(s^t)}{U_c(s^t)} = z(s_t)F_L(s^t), \qquad \forall s^t \in S^t; \tag{19}$$

and

$$c_t(s^t) + k_{t+1}(s^t) = z(s_t)F(k_t(s^{t-1}), \ell_t(s^t)) + (1 - \delta)k_t(s^{t-1}), \qquad \forall s^t \in S^t; \tag{20}$$

along with the following boundary conditions:

$$k_0 > 0 \quad \textit{given}, \quad \textit{and} \quad \lim_{t\to\infty} \beta^t \mathbb{E}\left[U_c(s^t)k_{t+1}(s^t)\right] = 0. \tag{21}$$

For any such sequence, the associated sequence of equilibrium prices

$$\left\{w_t(s^t), r_t(s^t), R_{t+1}(s^t)\right\}_{s^t \in S^t}$$

are given by:

$$r_t(s^t) = z(s^t)F_K(s^t), \qquad \forall s^t \in S^t \tag{22}$$

$$w_t(s^t) = z(s^t)F_L(s^t), \qquad \forall s^t \in S^t \tag{23}$$

and

$$U_c(s^t) = \beta(1 + R_{t+1}(s^t))\mathbb{E}_t\left[U_c(s^{t+1})\middle|\, s^t\right], \qquad \forall s^t \in S^t$$

Similar to the NGM, Proposition 2 has two parts, one about allocations (quantities) and another about equilibrium prices.

The first part says that conditions (18-21) are necessary and sufficient for an allocation to be part of an equilibrium. Note that these conditions do *not* involve prices: they permit us to test whether a candidate allocation is part of an equilibrium without looking at prices and market clearing! We henceforth refer to any allocation that satisfies (18-21) as an "equilibrium allocation."

The second part gives us the prices that "support" an equilibrium allocation. By this we mean the prices that have the following property: when the representative firm and the representative household face these prices, their optimal behavior admits the allocation under consideration.

4.4 Comparison between equilibrium allocations and the planner's solution

Consider Proposition 2 that characterize an equilibrium. Let us focus on conditions (18-21), which alone pin down an *equilibrium allocation*. We will now compare this to the conditions stated in Proposition 1 that characterize the solution to the planner's problem.

Condition (20) is simply the resource constraint for the economy: it describes the allocations that are technologically feasible. Note that it is the same as condition (6) in the planner's solution.

Condition (18) is the equilibrium Euler equation—also called the equilibrium intertemporal condition. It equates the marginal rate of substitution (MRS) between consumption today and tomorrow with the corresponding marginal rate of transformation (MRT). Note that it is the same as condition (4) in the planner's solution. In Proposition 2 this condition is interpreted as an *equilibrium* condition: it describes the joint optimality of the households, the firms, and market clearing. On the other hand, in Proposition 1, this condition is interpreted as the optimality condition of the social planner.

Condition (19) is the equilibrium intratemporal condition. It equates the marginal rate of substitution (MRS) between labor and consumption with the corresponding marginal rate of transformation (MRT). Note that it is the same as condition (5) in the planner's solution. In Proposition 2 this condition is interpreted as an *equilibrium* condition: it describes the joint optimality of the households, the firms, and market clearing. On the other hand, in Proposition 1, this condition is interpreted as the optimality condition of the social planner.

Finally, the boundary conditions in (21) coincide with the boundary conditions of the planner's solution (7).

Comparing the dynamic system of Proposition 2 with the one we had obtained for the planner's problem, Proposition 1, and noting that the planner's solution exists and is unique, we reach the following conclusion:

Theorem 1. *The competitive equilibrium allocation is unique and coincides with the planner's solution.*

We have therefore proved that the welfare theorems hold in the economy under consideration. These findings permit a direct reinterpretation of the optimal path characterized in Proposition 1 as the equilibrium path of the economy. The only novel applied lesson is that once we have the optimal sequence of consumption, labor, and capital from the planner's solution, we may obtain the model's predictions for the equilibrium rental rate and wage rate by reading it off the marginal products of capital and labor as in (22) and (23).

Again, as in the NGM, in the RBC model the incentives of the households and the firms are such that their joint optimality are aligned with the social planner's optimality. In other economies, such as those that feature externalities, monopoly power, or other types of distortions, this won't necessarily be the case, i.e. the planner's solution may not coincide with equilibrium allocations.

References

Cooley, Thomas F. and Edward C. Prescott, "Chapter 1: Economic Growth and Business Cycles," in "Frontiers of Business Cycle Research," Princeton University Press, 1995.

Kydland, Finn E. and Edward C. Prescott, "Time to Build and Aggregate Fluctuations," *Econometrica*, 1982, *50* (6), 1345–1370.

Stokey, Nancy L., Robert E. Lucas, and Edward C. Prescott, *Recursive Methods in Economic Dynamics*, Harvard University Press, 1989.