An Eventual Application of the Contraction Mapping Theorem

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Consider the planner's problem in the Neoclassical Growth Model, a.k.a. the Ramsey-Cass-Koopmans model. We will first write it down in the form of a *sequence problem*, but we will then reconsider it through the lens of *dynamic programming*. This lecture is similar to Chapter 2 of Stokey, Lucas and Prescott (1989).

1 The Neoclassical Growth Model

Consider the planner's problem in the Neoclassical Growth Model in discrete time. We assume time is discrete and indexed by $t = 0, 1, \ldots, \infty$. We write the planner's problem as follows.

Planner's Problem. Taking the initial condition as given:

$$k_0 > 0.$$

the social planner chooses an infinite sequence for consumption and capital,

$$\{c_t, k_{t+1}\}_{t=0}^{\infty},$$

so as to maximize the utility of the representative agent:

$$\sum_{t=0}^{\infty} \beta^t U(c_t) \tag{1}$$

with $\beta \in (0, 1)$, subject to the resource constraint,

$$c_t + k_{t+1} = f(k_t) + (1 - \delta)k_t, \qquad \forall t \geq 0 \tag{2}$$

and non-negativity constraints,

$$c_t \geq 0, \qquad k_{t+1} \geq 0, \qquad \forall t \geq 0.$$

We call this problem the *sequence problem*.

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Note that I am already writing the aggregate resource constraint with equality: if it were not satisfied at the planner's optimum with equality, then resources would have been left unused. The social planner could have raised social welfare simply by letting the household consume the unused resources.

Assumptions. By the Neoclassical assumptions on $F(K,L)$, we have that $f(k)$ is continuous, twice-differentiable, and satisfies:

$$f(0)=0, \qquad f'(k)>0, \qquad f''(k)<0$$

as well as the Inada conditions:

$$\lim_{k\to 0} f'(k)=\infty, \qquad \text{and} \qquad \lim_{k\to\infty} f'(k)=0.$$

For preferences, we make the typical regularity assumptions on U. That is, it is continuous, twice-differentiable, strictly increasing and strictly concave:

$$U'(c)>0, \qquad U''(c)<0$$

and satisfies the Inada conditions:

$$\lim_{c\to 0} U'(c)=\infty, \qquad \text{and} \qquad \lim_{c\to\infty} U'(c)=0.$$

2 Solution using the Lagrangian Method

One way of solving the sequence problem is to use the Lagrangian method. The main technicality one must be aware of, however, is that there are issues when dealing with infinite spaces.

By this I mean that what one typically does is set up the Lagrangian as if the horizon were finite: $t\in\{0,1,...T\}$ for some finite but large $T\geq 1$. One can then take a "hand-waving" limit as T approaches infinity. However, one needs to formally prove that this is the unique and correct solution to the planner's problem.

Let $T<\infty$. Letting $\beta^t\lambda_t$ denote the Lagrange multiplier on the period-t resource constraint (2), we have that the Lagrangian of the social planner's problem is given by:

$$\mathcal{L}=\sum_{t=0}^{T}\beta^t U(c_t)-\sum_{t=0}^{T}\beta^t\lambda_t\left[c_t+k_{t+1}-(1-\delta)k_t-f(k_t)\right]$$

We can then rewrite the Lagrangian as

$$\mathcal{L}=\sum_{t=0}^{T}\beta^t\left\{U(c_t)-\lambda_t\left[c_t+k_{t+1}-(1-\delta)k_t-f(k_t)\right]\right\}$$

Note that λ_t measures the planner's period-t shadow value of period-t resources—in short, it is the marginal social value of resources at time t.

One can henceforth assume an interior solution. As long as $k_t > 0$, an interior solution is indeed ensured by the Inada conditions on f and U. I will not characterize the solution to this problem in the interest of time, but you will probably do so in Xavier's class.

3 The Recursive Formulation

Instead in this class we will pursue another approach to solving this problem called dynamic programming.

Although we stated this problem as one of choosing infinite sequences for consumption and capital, $\{c_t, k_{t+1}\}_{t=0}^{\infty}$, starting from period zero, we can in fact restate the problem of the planner as one of choosing today's consumption c_0 and tomorrow's beginning of period capital k_1 and nothing else. The rest can wait until tomorrow. But the question is: what are the planner's preferences over current consumption and next period's capital?

The basic idea is to define a function that gives the value of next period's capital k_1. How would one do this? Suppose we had already solved the planner's problem stated above for all possible values of initial capital k_0. Then we could define a function

$$v : \mathbb{R}_+ \to \mathbb{R}$$

by letting $v(k_0)$ be the maximized objective function in (1) given $k_0 > 0$.

That is, suppose we solve this and get the optimal plan $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$, given some initial capital $k_0 > 0$. Then we define the function $v(k_0)$ to be

$$v(k_0) = \sum_{t=0}^{\infty} \beta^t U(c_t^*), \qquad \text{given } k_0 > 0.$$

That is, it $v(k_0)$ is the value of the maximized objective starting off with initial capital level $k_0 > 0$. We call this function the *value function*.

With the value function v so defined, $v(k_1)$ would then be the value of utility from period 1 on that could be obtained with a beginning-of-period capital stock k_1, and $\beta v(k_1)$ would be the value of this utility discounted back to period 0. Then in terms of this value function v, the planner's problem in period 0 could be written as follows:

$$\max_{c_0, k_1} [U(c_0) + \beta v(k_1)] \tag{3}$$

subject to resource and non-negativity constraints:

$$c_0 + k_1 = \gamma(k_0), \qquad c_0 \geq 0, \qquad k_1 \geq 0$$

and initial capital level $k_0 > 0$ given, where I have simply let the function γ denote the total goods available in the period:

$$\gamma(k_0) \equiv f(k_0) + (1 - \delta)k_0.$$

Note that I am using the fact that no goods are wasted.

At this point we do not "know" the function v but we have defined it as the maximized objective function for the problems stated in (1). Thus, if solving (3) provides the solution for that problem, then $v(k_0)$ must be the maximized objective function for (3) as well. That is, the function v must satisfy

$$v(k_0) = \max_{k_1 \in [0, \gamma(k_0)]} \left[U(\gamma(k_0) - k_1) + \beta v(k_1)\right]$$

where we have again used the fact that goods will not be wasted.

Note that when the problem is written in this recursive way, the time subscripts are unnecessary: the problem is the same at *every single date*. Thus, we can rewrite the problem facing the planner with current capital stock k as

$$v(k) = \max_{k' \in [0, \gamma(k)]} \left[U(\gamma(k) - k') + \beta v(k')\right] \tag{4}$$

where k' is used to denote capital taken into next period. Equation (4) is called the functional equation. It is also known as the Bellman equation, named after Richard Bellman (1957). The study of dynamic optimization problems through the analysis of Bellman equations is called dynamic programming.

As I've alluded to previously, even though this example is from growth theory, Bellman equations are regularly applied in dynamic settings in business cyle macro, industrial organization, structural labor, and finance. In order to establish the existence of the value function, one must use the Contraction Mapping Theorem (Lecture Notes 2), among other theorems. We are working towards understanding the Bellman equation and using this solution method in this class.

References

Stokey, Nancy L., Robert E. Lucas, and Edward C. Prescott, *Recursive Methods in Economic Dynamics*, Harvard University Press, 1989.