Optimization

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Almost all of economics is about optimization. In these lecture notes, we consider optimization problems in $\mathbb{R}^N$. These lecture notes draw on material found in Ok (2007), Sundaram (1996), and Rudin (1976).

1 Basic Topology on $\mathbb{R}^N$

Let's start with some basic topology. For our purposes, I will define open sets, closed sets, and compact sets on $\mathbb{R}^N$ with the Euclidean norm. For a more general treatment, please see Rudin (1976).

1.1 Open Sets

Definition 1. Let $X \subseteq \mathbb{R}^N$. X is *open* if for every $x \in X$ there exists an $\epsilon > 0$ such that

$$B_\epsilon(x) \subseteq X.$$

To make this more concrete, consider the following example of an open set.

Example: an open interval in $\mathbb{R}$. Let $X \equiv (a, b) \subset \mathbb{R}$. That is, let $a, b \in \mathbb{R}$ with $a < b$, and X is the open interval in $\mathbb{R}$ defined by:

$$X \equiv \{x \in \mathbb{R} | a < x < b\}.$$

To see that this is an open set in $\mathbb{R}$, we take any point $x \in (a, b)$ and define

$$\epsilon = \min\{x - a, b - x\} > 0.$$

By construction, $B_\epsilon(x) \subseteq X$.

Note that all open balls are open sets.

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1.2 Closed Sets

Definition 2. Let $X \subseteq \mathbb{R}^N$. X is *closed* if for every sequence $\{x_n\}_{n=0}^{\infty}$ in X that converges to $\bar{x} \in \mathbb{R}^N$,

$$\bar{x} \in X.$$

That is, a set is closed if every limit point of X is an element of X.

Theorem 1. *A set X is closed if and only if its complement X^c is open.*

Proof. First, Sufficiency. We first prove that if X^c is open, then X is closed.

Suppose that X^c is open. Consider any sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_n \in X$ for all n and converges to some $\bar{x} \in \mathbb{R}^N$. We need to show that $\bar{x} \in X$.

We prove this by contradiction. Suppose that $\bar{x} \in X^c$. Since X^c is open, there is some $\epsilon > 0$ for which

$$B_\epsilon(\bar{x}) \subseteq X^c.$$

Since $x_n \to \bar{x}$, there is an $N_\epsilon \in \mathbb{N}$ such that

$$\|x_n - \bar{x}\| < \epsilon.$$

for all $n \geq N_\epsilon$. Then for any $n \geq N_\epsilon$, we have that $x_n \in B_\epsilon(\bar{x}) \subseteq X^c$. That is, x_n is inside the open ball we constructed which is contained in X^c. But this is a contradiction. Therefore $\bar{x} \in X$.

Second, Necessity. We need to prove that if X is closed, then X^c is open. I will leave this part of the proof as an exercise. □

Exercise 1. Prove necessity: If X is closed, then its complement X^c is open.

Specifically, fix an element $x \in X^c$. One needs to show that there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq X^c$. The easiest way to prove this is by contradiction.

Consider the following example of a closed set.

Example: a closed interval in $\mathbb{R}$. Let $X \equiv [a, b] \subset \mathbb{R}$. That is, let $a, b \in \mathbb{R}$ with $a < b$, and X is the closed interval defined by:

$$X \equiv \{x \in \mathbb{R} | a \leq x \leq b\}.$$

To see that this is a closed set in $\mathbb{R}$, note that its complement,

$$X^c = (-\infty, a) \cup (b, \infty)$$

is open. We then apply Theorem 1.2.

1.3 Interior of Sets

Definition 3. Let $X \subseteq \mathbb{R}^N$. We say that a point x is an *interior point* of the set X if there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq X$. The set of all interior points of X is called the *interior of X* and is usually denoted by int(X).

Note that by definition int$(X) \subseteq X$.

1.4 Bounded Sets

Recall the definition of bounded sets in $\mathbb{R}$ from Lecture 1. Let me generalize this definition for sets in $\mathbb{R}^N$.

Definition 4. Let $X \subseteq \mathbb{R}^N$. X is *bounded* if there exists an $M > 0$ such that

$$X \subseteq B_M(0).$$

That is, X is bounded if it is completely contained in some open ball centered at the origin. Please note that this definition is only a slight generalization of the definition of boundedness that I gave you in lecture 1. There, for $X \subseteq \mathbb{R}$, X was said to be bounded if $|x| \leq M$ for all $x \in X$. Now, for $X \subseteq \mathbb{R}^N$, X is said to be bounded if $\|x\| \leq M$ for all $x \in X$ where $\|\cdot\|$ is the Euclidean norm.

1.5 Compact Sets

Definition 5. Let $X \subseteq \mathbb{R}^N$. X is *compact* if it is closed and bounded.

For the purists out there: I admit that this is a non-kosher way of defining compactness. Technically, the definition is the following: "Let (X, ρ) be metric space. $S \subseteq X$ is compact if every open cover of S contains a finite subcover," Rudin (1976), page 36. One then shows that a set $S \subset \mathbb{R}^N$ is compact if and only if it is closed and bounded; see Rudin (1976), page 40. In other words, the definition I provide above is a *result*.

In my opinion, the more general definition of compactness is unnecessarily complex for almost all applications of compactness in economics. I've never once had to use it (outside of my undergraduate math classes). Whereas, I feel as though I use the definition of compactness in $\mathbb{R}^N$ as being closed and bounded all the time!

For example, the closed interval $X \equiv [a, b] \subset \mathbb{R}$, is compact.

For the next result, we need to define the image of a set under a function.

Definition 6. Let $f : X \to Y$. Let $A \subseteq X$. The *image of A under f*, denoted by $f[A]$, is the set:

$$f[A] \equiv \{y \in Y | f(x) = y \text{ for some } x \in A\}.$$

In particular, the image $f[X]$ of the whole *domain* X is called the *range* of f.

Theorem 2. *Let $X \subseteq \mathbb{R}^N$ and let $f : X \to \mathbb{R}$ be continuous. Let $A \subseteq X$ be compact. The image of A under f, $f[A]$, is compact.*

Exercise 2. Prove Theorem 2.

In other words, the image of a compact set under a continuous function is compact.

2 Supremum and Infimum: a useful result

Let $X \subseteq \mathbb{R}$. Recall our definition of the sup and the inf from Lecture Notes 1, which I am repeating here to make our lives easier:

Definition 7. A real number $\alpha \in \mathbb{R}$ is the *least upper bound* of X, or the *supremum* of X, denoted by $\alpha = \sup X$, if it satisfies the following properties: (i) α is an upper bound of X, and, (ii) if $\gamma < \alpha$, then γ is not an upper bound of X.

Analogously, a real number $\beta \in \mathbb{R}$ is the *greatest lower bound* of X, or the *infimum* of X, denoted by $\beta = \inf X$, if it satisfies the following properties: (i) β is a lower bound of X and (ii) if $\gamma > \beta$ then γ is not an lower bound of X.

Let us state and prove the following useful result about the supremum.

Theorem 3. $\alpha = \sup X$ *if and only if for all* $\epsilon > 0$ *the following is true:*
(i) $x < \alpha + \epsilon$ *for all* $x \in X$*, and*
(ii) $x > \alpha - \epsilon$ *for some* $x \in X$.

Proof. First, Necessity. Let $\alpha = \sup X$ and $\epsilon > 0$ be arbitrary.

(i) Since α is an upper bound of X, then $x \leq \alpha$ for all $x \in X$. Therefore, $x < \alpha + \epsilon$ for all $x \in X$.

(ii) Suppose that the statement in part (ii) were not true. Then $x \leq \alpha - \epsilon$ for all $x \in X$. This would imply that $\alpha' = \alpha - \epsilon$ is an upper bound for X and $\alpha' < \alpha$. However this cannot be true if $\alpha = \sup X$ (by definition of the sup). This is a contradiction; we thus conclude that $\alpha - \epsilon < x$ for some $x \in X$.

Second, Sufficiency.

If for all $\epsilon > 0$, $x < \alpha + \epsilon$ for all $x \in X$, then $x \leq \alpha$ for all $x \in X$. This implies that α is an upper bound of X.

Suppose that α is not the least upper bound. Then there must exist another upper bound of X, call it α', such that $\alpha' < \alpha$. Let

$$\epsilon' \equiv \alpha - \alpha' > 0.$$

From part (ii) we know that there exists some $x \in X$ such that

$$x > \alpha - \epsilon' = \alpha'.$$

But this contradicts the statement that α' is an upper bound of X. Therefore α must be the least upper bound. □

Proposition 1. *Let* $X \subseteq \mathbb{R}$ *and suppose* $\alpha = \sup X$ *exists. There exists a sequence* $\{x_n\}_{n=0}^{\infty}$ *in* X *that converges to* α.

Exercise 3. Use Theorem 3 in order to prove Proposition 1. (This should be close to trivial. Proposition 1 is basically a corollary of Theorem 3.)

3 Maximum and Minimum

Let $X \subseteq \mathbb{R}$. Relative to the sup and the inf, we will also at times need a stronger concept of extrema, in particular one that implies that the extrema lie within the set.

Definition 8. A point $a \in \mathbb{R}$ is the *maximum* of a set $X \subseteq \mathbb{R}$, denoted by

$$a = \max X$$

if $a \in X$ and $x \leq a$ for all $x \in X$.

A point $b \in \mathbb{R}$ is the *minimum* of a set $X \subseteq \mathbb{R}$, denoted by

$$b = \min X$$

if $b \in X$ and $x \geq b$ for all $x \in X$.

The following two theorems are fairly intuitive.

Theorem 4. *If* $\max X$ *exists, then*
(i) it is unique, and
(ii) the $\sup X$ *exists and* $\sup X = \max X$.

Proof. Suppose the $\max X$ exists. Let $a = \max X$.

(i) We prove this by contradiction. Suppose a_1 and a_2 are both maxima of the set X and $a_1 \neq a_2$. Then $a_1, a_2 \in X$. By definition of the maxima:

$$a_1 \leq a_2, \qquad \text{and} \qquad a_2 \leq a_1.$$

But this implies that $a_1 = a_2$, a contradiction. We thus conclude that the max is unique.

(ii) Let α be an arbitrary upper bound of X. Since $a = \max X \in X$, then

$$\alpha \geq \max X.$$

Furthemore, $\max X$ is an upper bound of X. It follows that $\sup X$ exists and $\sup X = \max X$. □

Theorem 5. *If* $\sup X$ *exists and* $\sup X \in X$*, then* $\max X$ *exists and* $\max X = \sup X$.

Proof. Suppose $\sup X$ exists and $\sup X \in X$. Let $\alpha = \sup X$.

Then $\alpha \geq x$ for all $x \in X$ and $\alpha \in X$. Therefore, $\max X$ exists and $\max X = \alpha$. □

3.1 Extrema of functions

Typically, it is of more interest in economics to find extrema of functions rather than extrema of sets. To a large extent, there is no real distinction: looking for the maximum of a function over its domain is the same as looking for the maximum of the *image* of the domain under the function.

Definition 9. An element $\bar{x} \in X \subseteq \mathbb{R}^N$ is a *global maximizer* of $f : X \to \mathbb{R}$ if

$$f(x) \leq f(\bar{x})$$

for all $x \in X$.

If $\bar{x}$ is a global maximizer of f, then $f(\bar{x})$ is a *global maximum* of f. We typically write this as:

$$f(\bar{x}) = \max_{x \in X} f(x)$$

and

$$\bar{x} \in \arg\max_{x \in X} f(x)$$

Note that there is a conceptual difference between a maximum and a maximizer. Furthermore, a function can have at most one global maximum even if it has multiple global maximizers.

Definition 10. An element $\bar{x} \in X \subseteq \mathbb{R}^N$ is a *local maximizer* of $f : X \to \mathbb{R}$ if there exists some $\epsilon > 0$ such that

$$f(x) \leq f(\bar{x})$$

for all $x \in B_\epsilon(\bar{x}) \cap X$. If $\bar{x}$ is a local maximizer of f, then $f(\bar{x})$ is a *local maximum* of f.

4 Weierstrass Theorem: existence of the max and the min

Most of the time in economics, the problems we are interested in are optimization problems in which we maximize or minimize a function (often called the objective function) over a set.

We would like to have some conditions that guarantee that the solutions to such problems exist.

Theorem 6. The Weierstrass Theorem.

Let $X \subseteq \mathbb{R}^n$ be nonempty and compact. If the function $f : X \to \mathbb{R}$ is continuous, then f attains a maximum and minimum on X, i.e. there exists $x_h, x_\ell \in X$ such that

$$f(x_\ell) \leq f(x) \leq f(x_h)$$

for all $x \in X$.

Proof. Since X is compact and f is continuous, then the image of X under f, $f[X]$, is compact; Theorem 2.

Let $\alpha = \sup f[X]$; this is well-defined because $f[X]$ is bounded. By Proposition 1, there exists a sequence $\{y_n\}_{n=0}^{\infty}$ in $f[X]$ that converges to α.

Since $f[X]$ is compact, it is closed. Since it is closed, $\alpha = \sup f[X] \in f[X]$. Therefore, by Theorem 5, $\max f[X]$ exists and $\max f[X] = \sup f[X]$. Finally, there exists an $x_h \in X$ such that

$$f(x_h) = \max f[X],$$

and by definition of the max,

$$f(x) \leq f(x_h)$$

for all $x \in X$. The proof for the minimum is analogous. □

When the domain of a continuous function is compact, then the function attains a maximum and minimum on its domain.

The next exercise should make you think about what could potentially go wrong without each of those conditions.

Exercise 4. For the following examples, you can either write the sets and functions down in terms of mathematical notation or simply draw them:
(a) Provide an example of an unbounded set $X \subseteq \mathbb{R}$ and a continuous function $f : X \to \mathbb{R}$ in which a maximum of the function on its domain does not exist.
(b) Provide an example of a bounded but open set $X \subseteq \mathbb{R}$ and a continuous function $f : X \to \mathbb{R}$ in which a maximum of the function on its domain does not exist.
(c) Provide an example of a compact set $X \subseteq \mathbb{R}$ and a discontinuous function $f : X \to \mathbb{R}$ in which a maximum of the function on its domain does not exist.

It is important to note that the Weierstrass Theorem only gives us sufficient conditions for a maximum and minimum to exist. These conditions are by no means necessary. One can think of many examples in which these conditions do not hold, yet a maximum of the function exists.

References

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