Convex Structures in Optimization Theory

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In this lecture we study convex optimization problems. These lecture notes draw on material found in Chapter 7 of Sundaram (1996).

1 Convexity

1.1 Convex Sets

A set is convex if the line segment connecting any two elements x, x' in X are contained in X.

Definition 1. Let $X \subseteq \mathbb{R}^N$. X is *convex* if for all $x, x' \in X$ and $\theta \in [0,1]$,

$$\theta x + (1-\theta)x' \in X.$$

For example, open balls are convex.

1.2 Concave and Convex functions

Building on this definition of convex sets, we introduce two classes of functions called concave and convex functions.

Definition 2. Let X be a convex subset of $\mathbb{R}^N$. Let $f : X \to \mathbb{R}$.
(i) f is *concave* if for all $x, x' \in X$ and $\theta \in [0,1]$,

$$f(\theta x + (1-\theta)x') \geq \theta f(x) + (1-\theta)f(x').$$

(i) f is *strictly concave* if for all $x, x' \in X$, $x \neq x'$, and $\theta \in (0,1)$,

$$f(\theta x + (1-\theta)x') > \theta f(x) + (1-\theta)f(x').$$

It is worth mentioning that the domain of f must be a convex set in order for f to be a concave function. Otherwise, the idea of concavity is undefined.

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Definition 3. Let X be a convex subset of $\mathbb{R}^N$. Let $f : X \to \mathbb{R}$.

(i) f is *convex* if for all $x, x' \in X$ and $\theta \in [0, 1]$,

$$f(\theta x + (1-\theta)x') \leq \theta f(x) + (1-\theta)f(x').$$

(i) f is *strictly concave* if for all $x, x' \in X$, $x \neq x'$, and $\theta \in (0, 1)$,

$$f(\theta x + (1-\theta)x') < \theta f(x) + (1-\theta)f(x').$$

Note that the notions of concavity and convexity are neither exhaustive nor mutually exclusive; that is, there are functions that are neither concave nor convex, and functions that are both concave and convex. For example, linear (affine) functions are both concave and convex.

Exercise 1. (a) Show that the linear function $f : \mathbb{R}^N \to \mathbb{R}$ defined by $f(x) = a \cdot x + b$, with $a \in \mathbb{R}^N$and $b \in \mathbb{R}$ is both concave and convex.

(b) Conversely, show that if $f : \mathbb{R}^N \to \mathbb{R}$ is both convex and concave, then it is a linear function.

2 Implications of Convexity

Recall our definitions of global and local maxima from the previous lecture.

Definition 4. An element $\bar{x} \in X \subseteq \mathbb{R}^N$ is a *global maximizer* of $f : X \to \mathbb{R}$ if

$$f(x) \leq f(\bar{x})$$

for all $x \in X$. If $\bar{x}$ is a global maximizer of f, then $f(\bar{x})$ is a *global maximum* of f.

Definition 5. An element $\bar{x} \in X \subseteq \mathbb{R}^N$ is a *local maximizer* of $f : X \to \mathbb{R}$ if there exists some $\epsilon > 0$ such that

$$f(x) \leq f(\bar{x})$$

for all $x \in B_\epsilon(\bar{x}) \cap X$. If $\bar{x}$ is a local maximizer of f, then $f(\bar{x})$ is a *local maximum* of f.

In economics we are often interested in problems of the following form:

$$\max_{x \in X} f(x).$$

where X is the constraint set and f is the objective function.

Definition 6. We refer to a maximization problem as a *convex maximization problem* if the constraint set is convex and the objective function is concave.

Similarly, we refer to a minimization problem as a *convex minimization problem* if the constraint set is convex and the objective function is convex.

More generally, we refer to an optimization problem as a *convex optimization problem* if it is either of the above.

We will now study some general results of convex optimization. All results are stated in the context of convex maximization problems, but each has an exact analogue in the context of convex minimization problems.

Theorem 1. *Let $X \subset \mathbb{R}^N$ be convex, and let $f : X \to \mathbb{R}$ be concave. If x_1 is a local maximizer of f on X, then it is also a global maximizer.*

Proof. Since x_1 is a local maximizer of f, there exists an $\epsilon > 0$ such that $f(x_1) \geq f(x)$ for all $x \in B_\epsilon(x_1) \cap X$.

Suppose x_1 is not a global maximizer. Then there exists an $x_2 \in X$ such that

$$f(x_2) > f(x_1).$$

Clearly, $x_2 \notin B_\epsilon(x_1)$.

Since X is convex,

$$\theta x_1 + (1-\theta)x_2 \in X$$

for all $\theta \in (0,1)$.

Pick θ sufficiently close to 1 so that $z = \theta x_1 + (1-\theta)x_2 \in B_\epsilon(x_1)$. By concavity of f

$$f(z) \geq \theta f(x_1) + (1-\theta)f(x_2) > f(x_1)$$

where the second inequality is due to $f(x_2) > f(x_1)$. Thus $f(z) > f(x_1)$. But $z \in B_\epsilon(x_1)$ by construction, so this is a contradiction. $\square$

Therefore, if a strictly concave function has a local maximum, then that point is a global maximum.

Next we establish some results on the set of global maximizers.

Theorem 2. *Let $X \subset \mathbb{R}^N$ be convex, and let $f : X \to \mathbb{R}$ be concave.*

Then the set $\arg\max\{f(x) | x \in X\}$ of maximizers of f on X is either empty or convex.

Proof. Suppose $\arg\max\{f(x) | x \in X\}$ is nonempty. We will show that it must be convex.

Suppose x_1 and x_2 are both maximizers of f on X. Then, we have $f(x_1) = f(x_2)$.

By concavity, for any $\theta \in (0,1)$ we have:

$$f(\theta x_1 + (1-\theta)x_2) \geq \theta f(x_1) + (1-\theta)f(x_2) = f(x_1).$$

This must hold with equality or else x_1 and x_2 would not be maximizers. Thus the set of maximizers contains $\theta x_1 + (1-\theta)x_2$ for any $\theta \in (0,1)$ and therefore must be convex. $\square$

In other words, in convex optimization problems, we cannot have multiple isolated points as maximizers.

Finally, we show that a strictly concave objective function has at most one unique maximizer.

Theorem 3. *Let $X \subset \mathbb{R}^N$ be convex, and let $f : X \to \mathbb{R}$ be strictly concave.*

The set $\arg\max\{f(x) | x \in X\}$ of maximizers of f on X is either empty or contains a single point.

Proof. Suppose $\arg\max\{f(x)|x \in X\}$ is nonempty. Let x_1 be a maximizer of f on X. We will show that there can be no other maximizers.

Suppose x_2 is also a maximizer f on X and $x_2 \neq x_1$. Then $f(x_1) = f(x_2)$.

By strict concavity, for any $\theta \in (0, 1)$ we have:

$$f(\theta x_1 + (1-\theta)x_2) > \theta f(x_1) + (1-\theta)f(x_2) = f(x_1).$$

But this is a contradiction of x_1 being a maximizer. □

Conclusion. To conclude, in convex optimization problems, all local optima must also be global optima. Therefore in order to find a global maximum it suffices to find a local maximum.

Furthermore, if the objective function is strictly concave, then if the problem admits a solution, the solution is unique.

References

Sundaram, Rangarajan K., *A First Course in Optimization Theory*, Cambridge University Press, 1996.