# Early Extensions of the RBC

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In the previous lectures we studied the basic, bare-bones RBC model and explored its empirical implications. In this lecture we consider two early extensions of the RBC which help reconcile the model with some of discrepancies in the data. In particular we study: (i) indivisible labor and lotteries, and (ii) variable capital utilization.

## 1 Indivisible Labor and Lotteries

An early criticism of the RBC model was that it is unable to match the observed volatility of labor over the business cycle. This is because the standard RBC model with a representative household with only an intensive-margin of labor supply would require relatively high elasticities of labor supply to match the observed high variability of hours worked together with the low variability of real wage rates. Exercises by MaCurdy (1981) and Altonji (1986) demonstrated that this could not be possible with the standard estimates of labor supply elasticities from the micro data, which were much smaller than required by the macro data. Some claimed that this inconsistency was evidence against the entire RBC approach; see e.g. Summers (1986).

Heckman (1984) (in a discussion of Ashenfelter and Kydland) instead suggested that the key underlying issue was that models being used in both micro and macro were abstracting from the extensive margin of labor adjustment, and that this omission was likely important. Shortly after Heckman's discussion, macroeconomists found a way to tractably introduce an extensive margin of labor supply into models. Rogerson (1988) was the first to introduce an indivisible labor assumption and model an extensive margin of labor adjustment. Building on the theoretical insights of Rogerson (1988), Hansen (1985) introduced the indivisible labor assumption into an otherwise standard RBC model.[1]

### 1.1 Setup without lotteries

Rogerson (1988) introduces indivisible labor supply. Consider a simple static model (single time period). There is a continuum of identical agents, indexed by $i \sim U[0,1]$. The representative firm's production function is given by

$$Y = F(K, L)$$

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[1]This tidbit of history is taken from Rogerson (2011).

Each agent's household utility is given by

$$u(c) - v(\ell)$$

where $c \geq 0$ and $\ell \in \{0, 1\}$ is *indivisible*. We assume

$$\begin{aligned} v(0) &= 0 \\ v(1) &= m > 0 \end{aligned}$$

We assume indivisible labor: individuals can either work some given positive number of hours or not work at all; they are unable to work an intermediate number of hours. In other words, labor may only adjust along the extensive margin and cannot adjust along the intensive margin. This assumption is motivated by the observation that most people either work full time or not at all.

Each household is endowed with 1 unit of capital which it chooses to rent out to the firms. We define the feasibility set for each household as

$$X = \{c, \ell, k | c \geq 0, \ell \in \{0, 1\}, k \in [0, 1]\} \tag{1}$$

The household's problem is thus to maximize

$$\max_{c,l,k \in X} u(c^i) - m\ell^i$$

subject to

$$c^i \leq w\ell^i + rk^i.$$

Note that the household will clearly choose $k = 1$.[2]

There is a representative firm with technology $Y = F(K, L)$.

**Definition 1.** A competitive equilibrium is a set of allocations and prices such that

(i) households maximize utility subject to their budget sets and feasibility set (1).

(ii) firms maximize profits,

$$\max_{K \geq 0, L \geq 0} F(K, L) - rK - wL,$$

(iii) and markets clear

$$K = \int k_i di = 1, \qquad L = \int \ell_i di, \qquad \int c_i di = Y = F(K, L)$$

Note that the feability set for labor is a non-convex set. As a result, the second welfare theorem cannot be applied. That is, the Pareto optimal allocation in this environment cannot necessarily be decentralized as a competitive equilibrium.

[2]The only reason Rogerson includes capital is to allow for a CRS technology together with diminishing marginal product of labor.

## 1.2 Setup with Lotteries

Rogerson (1988) further shows that lotteries are necessary to implement Pareto optimal allocations in this environment. This in turn implies an *aggregate* labor supply that is infinitely elastic with respect to the (real) wage.

Consider the same environment as above, except that the feasible set is now expanded by introducing a specific class of lotteries. Consider the following lottery. An element in the household's consumption set may now be written as

$$\{(c_1, 1, 1), (c_2, 0, 1), \phi\} \in \bar{X}$$

which describes the following lottery:

- with probability $\phi$, the agent consumes $c_1$, supplies $1$ unit of labor (is employed), and supplies capital $k_1 = 1$.
- with probability $1 - \phi$, the agent consumes $c_2$, supplies $0$ units of labor (is unemployed), and supplies capital $k_2 = 1$.

This lottery essentially convexifies the feasibility set.

Furthermore, we assume agents can purchase insurance for this income uncertainty.[3] The household's budget constraints must satisfy

$$\begin{aligned} c_1 &\leq w + r - x_1 \\ c_2 &\leq r + x_2 \end{aligned}$$

where $x_1$ is the premium they pay the insurer in the employed state and $x_2$ is the amount they receive from the insurer in the unemployed state. Actuarially fair insurance (insurers make zero profit) implies

$$\pi = \phi x_1 - (1 - \phi)x_2 = 0.$$

Substituting for $x_1$ and $x_2$ from the budget contraints into this expression yields

$$\phi(c_1 - w - r) - (1 - \phi)(r - c_2) = 0$$

Therefore, the budget constraint of the household reduces to

$$\phi c_1 + (1 - \phi)c_2 = \phi w + r$$

As a result, we may write the household's problem as follows

$$\max_{c_1, c_2, \phi} \phi(u(c_1) - m) + (1 - \phi)u(c_2)$$

[3] Equivalently there are Arrow-Debreu securities contingent upon individual outcomes of the lottery. In other words, markets are complete.

subject to

$$\phi c_1 + (1-\phi)c_2 = \phi w + r$$

and

$$c_1 \geq 0, \qquad c_2 \geq 0, \qquad \phi \in [0,1].$$

The firm's problem clearly remains the same. Market clearing is given by:

$$\begin{aligned} K &= \phi k_1 + (1-\phi)k_2 \\ L &= \phi \\ F(K,L) &= \phi c_1 + (1-\phi)c_2 \end{aligned}$$

**Lemma 1.** *The household's optimal plan will have* $c_1 = c_2$.

*Proof.* FOCs to household's problem are given by

$$\begin{aligned} \phi u'(c_1) - \lambda\phi &= 0 \\ (1-\phi)u'(c_2) - \lambda(1-\phi) &= 0 \end{aligned}$$

Thus, $c_1 = c_2$. □

The household fully insures. (Alternatively, you could think of the representative household as a "big family" that decides how many of its members to send to work, but all members consume the same amount.) Using this optimal consumption plan along with the fact that $k_1 = k_2 = 1$ we reduce the competitive equilibrium of this economy to the following.

**Definition 2.** A competitive equilibrium is a set of allocations and prices such that

(i) households maximize utility

$$\max_{c,\phi} u(c) - m\phi$$

subject to

$$c = w\phi + r$$

$c \geq 0$ and $\phi \in [0,1]$.

(ii) firms maximize profits

$$\max_{K\geq 0, L\geq 0} F(K,L) - rK - wL,$$

(iii) and markets clear

$$K = 1, \qquad L = \phi, \qquad c = Y = F(K,L).$$

### 1.3 A Useful Isomorphism

Note that the above equilibrium is identical to the one that would obtain for an economy with technology

$$F(1,L)$$

and a representative agent with utility given by

$$u(C) - mL$$

with

$$C \geq 0, \qquad \text{and} \qquad 0 \leq L \leq 1$$

This economy has zero non-convexities. Therefore, one can then define the planners problem for this isomorphic economy as the following.

**The Planner's Problem.**

$$\max_{c,\ell} u(C) - mL$$

subject to

$$C \leq F(1, L)$$

The solution to this planner's problem is identical to the competitive equilibrium allocation in the economy with indivisible labor and lotteries.

### 1.4 Implications for Aggregate Fluctuations

Why is this idea useful? As we said, one of the earliest criticisms of the RBC model is its inability to account for observed relative magnitudes of fluctuations in total labor supply. In order to generate large fluctuations in labor, the model must be calibrated with a high elasticity of labor supply, around 2-4. However, microeconomic studies at the time had already estimated that the elasticity of labor supply is rather low, around 0 to .5. Thus estimates of the elasticity of labor supply using micro data are much smaller than that required to reconcile RBC models with the data on aggregate fluctuations.

The Rogerson (1988) model of individisible labor and lotteries is a new mechanism that allows a high aggregate elasticity of labor supply to be compatible with low labor supply elasticities for individual workers. Hansen (1985), building off of Rogerson (1988)'s theoretical insight, applied this idea to the otherwise standard RBC and explored its quantitative implications.

**Parenthesis. The Elasticity of Labor Supply.** Consider the representative household (as in the original RBC model) and suppose the household's preferences are additively-seperable and given by

$$u(c) - \frac{\ell^{1+\epsilon}}{1+\epsilon}.$$

In this case, the household's intratemporal condition is

$$\frac{\ell^{\epsilon}}{u'(c)} = w.$$

That is, the marginal rate of substitution between consumption and labor is equal to the wage rate. Abstracting from the income effect, we may write

$$\log \ell = \frac{1}{\epsilon} \log w + \cdots$$

Thus it is easy to see that the Frisch elasticity of labor supply is $\frac{1}{\epsilon}$. That is, $1/\epsilon$ parameterizes how labor supply responds to the wage (holding the wealth effect constant). Note that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} = \infty$$

Thus, a utility function that is linear in labor has an *infinite* elasticity of labor supply. However, estimates from micro data suggest an extremely low elasticity of labor supply; Altonji (1986).

**Reconciling the RBC with the data.** Indivisibilities (i.e. non-convexities) in labor and lotteries may help reconcile the RBC with the data. Note that in the original economy, the utility function of the individual household was

$$u(c) - v(\ell)$$

with $\ell \in \{0, 1\}$. However, we have just proven that this economy behaves as though there is a single representative agent with preferences given by

$$u(c) - m\ell$$

These preferences are quasilinear in labor, and hence has infinite elasticity of labor supply. There is thus a discrepancy between the true preferences of agents, $v(\ell)$, which could have been parameterized with a low elasticity of labor, and the preferences of the hypothetical representative household generating aggregate fluctuations.

A very high macro elasticity of labor supply may be independent of the labor supply elasticity of individual workers. This property results from the fact that in the model all variation in hours worked comes from the extensive margin, i.e., from workers moving in and out of the labor force. The elasticity of labor supply of an individual worker, i.e. the answer to the question "if your wage increased by one percent, how many more hours would you choose to work?," is irrelevant, because the number of hours worked is not a choice variable.

As a result, the model with indivisibilities in labor behaves as if there were a representative household with infinite elasticity of labor. With infinite elasticity of labor, the RBC model can generate a huge response of aggregate labor to an innovation in productivity. See the following figure from Hansen (1985).

**Remarks.** The standard RBC model adopts a rudimentary description of the labor market: firms hire workers in competitive spot labor markets and there is no unemployment. The Hansen (1985)-Rogerson (1988) extension with indivisibilities of labor supply does in fact generate unemployment. However, one unattractive feature of the model is that participation in

*G.D. Hansen, Indivisible labor and the business cycle* 321

**Table 1**

**Standard deviations in percent (a) and correlations with output (b) for U.S. and artificial economies.**

| | Quarterly U.S. time series[a] (55.3–84.1) | | Economy with divisible labor[b] | | Economy with indivisible labor[b] | |
|---|---|---|---|---|---|---|
| Series | (a) | (b) | (a) | (b) | (a) | (b) |
| Output | 1.76 | 1.00 | 1.35 (0.16) | 1.00 (0.00) | 1.76 (0.21) | 1.00 (0.00) |
| Consumption | 1.29 | 0.85 | 0.42 (0.06) | 0.89 (0.03) | 0.51 (0.08) | 0.87 (0.04) |
| Investment | 8.60 | 0.92 | 4.24 (0.51) | 0.99 (0.00) | 5.71 (0.70) | 0.99 (0.00) |
| Capital stock | 0.63 | 0.04 | 0.36 (0.07) | 0.06 (0.07) | 0.47 (0.10) | 0.05 (0.07) |
| Hours | 1.66 | 0.76 | 0.70 (0.08) | 0.98 (0.01) | 1.35 (0.16) | 0.98 (0.01) |
| Productivity | 1.18 | 0.42 | 0.68 (0.08) | 0.98 (0.01) | 0.50 (0.07) | 0.87 (0.03) |

the labor force is dictated by a lottery that makes the choice between working and not working convex.

An important research topic at the interface between macroeconomics and labor economics is understanding the role of wages and the dynamics of unemployment. Search and matching models—referred to as the Diamond-Mortensen-Pissarides Model—have emerged as a framework that is suitable for understanding not only the dynamics of unemployment, but also the properties of vacancies and of flows in and out of the labor force. Diamond, Mortensen, and Pissarides were awarded the Nobel prize for their work in 2010. See the original work by Diamond (1982), Mortensen (1982), Pissarides (1990), and Mortensen and Pissarides (1994).

## 2 Capital Utilization

The RBC models short-run fluctuations as movements in the economy's production function–booms and busts are due to exogenous shocks in TFP. Given the model, the Solow residual is supposed to measure these movements in technology. However, the Solow residual is simply a residual—it is not a true measure of productivity; in particular the following measurement issues should be considered:

- Variation in the Solow residual probably captures variation in various inputs beyond the standard $K$ and $L$. e.g., energy, materials, capital utilization, etc.
- The Solow residual can be predicted by military spending, monetary policies, etc. (e.g., Hall, 1988, Evans, 1992)
- The Solow residual implies implausibly high probability of technological regress. Remember, recessions are caused by negative productivity shocks.
- Plant-level measurements of productivity suggest much smaller volatility (e.g., Basu and Kimball, 1997)
- Proxies of capital utilization such as electricity consumption or proxies of labor hoarding/effort such as accidents and time-use are highly procyclical (e.g., Burnside, Eichenbaum and Rebelo, 1993, 1996; Basu and Kimball, 1997)

Thus, one possible resolution to these issues is the following. By adding variation in capital utilization, labor hoarding or other unmeasured inputs we can amplify the impact of small productivity shocks, leading to highly volatile Solow residuals.

We proceed by adding a new input into the technology:

$$y(s^t) = z(s_t)F(x(s^t)k_t(s^{t-1}), \ell(s^t))$$

where $z(s_t)$ is the "true" TFP shock and $x(s^t)$ represents endogenous capital utilization.

The cost of higher capital utilization is faster depreciation:

$$k_{t+1}(s^t) = [1 - \delta(x(s^t))]k_t(s^{t-1}) + i(s^t)$$

where the depreciatioin function, $\delta(x)$, is increasing and convex in utilization:

$$\delta', \delta'' > 0.$$

The resource constraint is therefore given by

$$c(s^t) + k_{t+1}(s^t) = [1 - \delta(x(s^t))]k_t(s^{t-1}) + z(s_t)F(x(s^t)k_t(s^{t-1}), \ell(s^t))$$

In this case, the optimality (or equilibrium) condition for capital utilization is given by

$$z(s_t)F_K(s^t)k_t(s^{t-1}) = \delta'(x(s^t))k_t(s^{t-1})$$

which reduces to

$$z(s_t)F_K(s^t) = \delta'(x(s^t)) \tag{2}$$

Next, suppose we specify the following Cobb-Douglas production function and homothetic depreciation function:

$$F(xk, \ell) = (xk)^{\alpha}\ell^{1-\alpha}$$

$$\delta(x) = \mu\frac{x^{1+\xi}}{1+\xi}.$$

for some $\alpha \in (0,1)$, $\xi > 0$, and constant $\mu > 0$. Note that $\xi \to \infty$ is the benchmark case with no movement in capital utilization, whereas $\xi \to 0$ is the opposite extreme of completely linear depreciation in utilization.

Then, using these functional forms in our optimality condition (2) we get.

$$\alpha\frac{y(s^t)}{x(s^t)k_t(s^{t-1})} = \mu x(s^t)^{\xi}$$

Next, we normalize $\mu = \alpha$ (as $\mu$ is just a constant). Solving for capital utilization we get:

$$x(s^t) = \left(\frac{y(s^t)}{k_t(s^{t-1})}\right)^{1/(1+\xi)}$$

Plugging this into the production function gives us

$$y(s^t) = z(s_t)^{\frac{1+\xi}{1-\alpha+\xi}}k_t(s^{t-1})^{\frac{\alpha\xi}{1-\alpha+\xi}}\ell(s^t)^{1-\frac{\alpha\xi}{1-\alpha+\xi}}$$

This leads us to the following reduced-form production function:

$$y(s^t) = z(s_t)^{\eta}k_t(s^{t-1})^{\hat{\alpha}}\ell(s^t)^{1-\hat{\alpha}}$$

where

$$\eta \equiv \frac{1+\xi}{1-\alpha+\xi} > 1 \qquad \text{and} \qquad \hat{\alpha} \equiv \frac{\alpha\xi}{1-\alpha+\xi} < \alpha.$$

Note that:

$$\frac{\partial\eta}{\partial\xi} < 0 \qquad \frac{\partial\hat{\alpha}}{\partial\xi} > 0$$

That is, in the benchmark with no capital utilization, $\xi \to \infty$, we have:

$$\eta \to 1 \qquad \text{and} \qquad \hat{\alpha} \to \alpha$$

Whereas in the limit with linear depreciation, $\xi \to 0$, we have:

$$\eta \to \frac{1}{1-\alpha} \qquad \text{and} \qquad \hat{\alpha} \to 0$$

In this limit, the effect of productivity is amplified, and production becomes almost linear in labor.

Why is this important? The Solow residual mismeasures true productivity; true productivity can be much less volatile than measured Solow residual. We can thus recalibrate the RBC model with lower volatility in productivity. A lower $\xi$ implies higher volatility in the Solow residual for any given volatility in true TFP, $z(s_t)$, which in turn generates larger fluctuations from smaller primitive shocks.

Note that there is another indirect effect of this mechanism: a lower $\xi$ implies a lower $\hat{\alpha}$. Thus, the demand for labor becomes more elastic in response to the wage. The equilibrium conditions for labor are given by:

$$(1-\hat{\alpha})z(s_t)^{\eta}k_t(s^{t-1})^{\hat{\alpha}}\ell(s^t)^{-\hat{\alpha}} = w(s^t) = \frac{\ell(s^t)^{\epsilon}}{u'(c(s^t))},$$

where I have assumed homothetic disutility of labor of the household:

$$v(\ell) = \frac{\ell(s^t)^{1+\epsilon}}{1+\epsilon}$$

with $\epsilon > 0$. Solving this for labor and writing in logs we get:

$$\log \ell(s^t) = \frac{1}{\epsilon+\hat{\alpha}}\left[\eta \log z(s_t) + \cdots\right]$$

A lower $\xi$ implies both a lower $\hat{\alpha}$ and a higher $\eta$. As a result, equilibrium employment responds more strongly to movements in true productivity $z(s_t)$.

Therefore, with variable capital utilization, output responds more to true productivity for two reasons: both the direct effect and an indirect effect through labor demand.

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