Theorem of the Maximum

Jennifer La'O*

These lecture notes draw on material found in Chapter 3 Stokey, Lucas and Prescott (1989) and Chapter 9 of Sundaram (1996).

1 Problems we are interested in

Recall from Lecture Notes 1 that a *correspondence* Γ from a set X into a set Y, denoted $\Gamma : X \rightarrow Y$, is a rule that assigns to each $x \in X$ a set $\Gamma(x) \subset Y$.

Let $X \subseteq \mathbb{R}^n$ and let $Y \subseteq \mathbb{R}^m$ and let

$$\varphi : X \times Y \rightarrow \mathbb{R}$$

be a (single-valued) function and let

$$\Gamma : X \rightarrow Y$$

be a non-empty correspondence. Our interest is in problems of the form

$$\sup_{y \in \Gamma(x)} \varphi(x, y).$$

where φ is the objective function, y is the "choice" or "control" variable, x is the state variable, and Γ describes the constraint set for y given x. Intuitively one should think of the state variable as anything that the decision maker takes as given when making their decision. On the other hand, the decision maker has control over the choice variable y.

Example. A standard example is in consumer theory. One can think of $y = c = (c_1, c_2, \ldots, c_M)$ as a vector of M consumption goods, i.e. a consumption bundle, and $x = (p, I)$ as a vector of prices for these goods, $p = (p_1, p_2, \ldots, p_M)$ and the income I of the consumer. In this case the objective function would be the utility from consumption: $\varphi(x, y) = U(y) = U(c)$. The constraint set $\Gamma(x) = \Gamma(p, I)$ would denote the consumer's budget set given prices p and income level I. Specifically:

$$\Gamma(p, I) = \{c \in \mathbb{R}^M_+ | p \cdot c \leq I\}. \tag{1}$$

*Department of Economics, Columbia University, jenlao@columbia.edu.

1.1 Application of Weierstrass

If for each $x \in X$, $\varphi(x, \cdot)$ is continuous in y and the set $\Gamma(x)$ is nonempty and compact, then thanks to the Weierstrass Theorem for each x the maximum is attained. In this case the function

$$h(x) = \max_{y \in \Gamma(x)} \varphi(x, y)$$

is well-defined, and the set of maximizers

$$G(x) = \{y \in \Gamma(x) | \varphi(x, y) = h(x)\},$$

is non-empty. We say that $G(x)$ is the set of y values that solve the problem given x, and $h(x)$ is the maximum.

Example. Going back to our consumer theory example, one typically assumes that the utility function U is continuous. It is trival that the budget set $\Gamma(p, I)$ defined in (1) is nonempty and compact (it's clearly bounded and its complement is open). Therefore the function

$$v(p, I) = \max_{c \in \Gamma(p,I)} U(c)$$

is well-defined, and

$$c^*(p, I) = \{c \in \Gamma(p, I) | U(c) = v(p, I)\}$$

is non-empty. In consumer theory we typically call $v(p, I)$ the indirect utility function and $c^*(p, I)$ the demand correspondence (and when single-valued we call it the demand function).

1.2 The question

We are interested in the following question: under what conditions do $h(x)$ and $G(x)$ vary continuously with the state x? At an intuitive level, one would think that in order to obtain continuity of the solution to the maximization problem, one would need some degree of continuity of the primitives, φ and Γ, of the problem.

We've already studied to a certain extent the continuity of functions, but we have not yet considered what it means for a correspondence to be continuous. We do this next.

2 Upper and Lower Hemi-Continuity

Let $X \subseteq \mathbb{R}^N$ and let $Y \subseteq \mathbb{R}^M$ with the Euclidean norms.

Definition 1. A correspondence $\Gamma : X \to Y$ is said to be:
(i) *closed-valued at* $x \in X$ if $\Gamma(x)$ is a closed set,
(ii) *compact-valued at* $x \in X$ if $\Gamma(x)$ is a compact set,
(iii) *convex-valued at* $x \in X$ if $\Gamma(x)$ is a convex set.

A correspondence is *closed-valued* if it is closed-valued at every point $x \in X$, it is *compact-valued* if it is compact-valued at every point $x \in X$, and it is *convex-valued* if it is convex-valued at every point $x \in X$.

Exercise 1. Consider the correspondence $\Gamma(p, I)$ defined in (1) that represents the consumer's budget set in our consumer theory example. We have already established that $\Gamma(p, I)$ is nonempty and compact-valued. Show that it is also convex-valued.

We now define two notions of continuity.

Definition 2. A correspondence $\Gamma : X \to Y$ is *lower hemi-continuous (l.h.c.) at* $x \in X$ if, for every $y \in \Gamma(x)$ and every sequence $x_n \to x$ there exists $N \geq 1$ and a sequence $\{y_n\}_{n=N}^{\infty}$ such that

$$y_n \to y \qquad \text{and} \qquad y_n \in \Gamma(x_n),\ \forall n \geq N.$$

Definition 3. A compact-valued correspondence $\Gamma : X \to Y$ is *upper hemi-continuous (u.h.c.) at* $x \in X$ if $\Gamma(x)$ is nonempty and if, for every sequence $x_n \to x$ and every sequence $\{y_n\}_{n=1}^{\infty}$ such that $y_n \in \Gamma(x_n)$ for all $n \geq 1$, there exists a convergent subsequence of $\{y_n\}$ whose limit point y is in $\Gamma(x)$.

Definition 4. A correspondence $\Gamma : X \to Y$ is *continuous at* $x \in X$ if it is both u.h.c. and l.h.c. at x. A correspondence is *continuous* if it is continuous at every point $x \in X$.

In order to visualize these definitions, consider Figure 1. Figure 1 displays a correspondence that is l.h.c. but not u.h.c. at x_1 and is u.h.c. but not l.h.c. at x_2. The correspondence in the figure is both u.h.c. and l.h.c. at all other points.

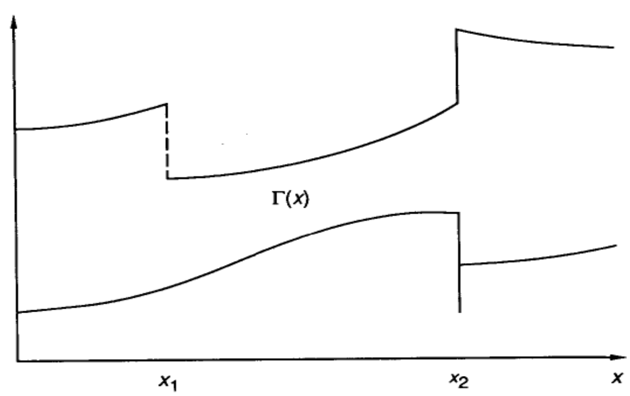

Figure 1. An illustration of u.h.c. and l.h.c.

Exercise 2. Show that if Γ is single-valued and u.h.c., then it is a continuous function.

Exercise 3. (a) Let $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $\Gamma(x) = [0, x]$. Show that Γ is continuous.

(b) Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function. Define the correspondence $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ by $\Gamma(x) = [0, f(x)]$. Show that Γ is continuous.

3 Theorem of the Maximum

We are now ready to answer under what conditions do $h(x)$ and $G(x)$ vary continuously with x. The following theorem is called the Theorem of the Maximum.

Theorem 1. *(*The Theorem of the Maximum.*) Let $X \subseteq \mathbb{R}^N$ and let $Y \subseteq \mathbb{R}^M$.*

Let $\varphi : X \times Y \to \mathbb{R}$ be a continuous function, and let $\Gamma : X \to Y$ be a nonempty, compact-valued, and continuous correspondence.

Then the function $h : X \to \mathbb{R}$ defined by:

$$h(x) = \max_{y \in \Gamma(x)} \varphi(x, y) \tag{2}$$

is continuous, and the correspondence $G : X \to Y$ defined by:

$$G(x) = \{y \in \Gamma(x) | \varphi(x, y) = h(x)\}. \tag{3}$$

is non-empty, compact-valued, and upper-hemi-continuous.

Proof. Note that for each $x \in X$, the set $\Gamma(x)$ is nonempty and compact and the function $\varphi(x, \cdot)$ is continuous in y. By the Weierstrass Theorem the maximum is attained and the set $G(x)$ of maximizers is nonempty.

Next $G(x) \subseteq \Gamma(x)$ and $\Gamma(x)$ is compact. It follows that $G(x)$ is bounded.

We next show that $G(x)$ is closed. Consider a sequence $\{y_n\}$ in that converges to $y \in Y$ and $y_n \in G(x)$ for all n. Since $\Gamma(x)$ is closed, $y \in \Gamma(x)$. Also since $h(x) = \varphi(x, y_n)$ for all n and φ is continuous in y, it follows that $\varphi(x, y) = h(x)$. But this implies that $y \in G(x)$. Hence $G(x)$ is closed. $G(x)$ is closed and bounded, therefore $G(x)$ is compact.

We next show that $G(x)$ is u.h.c. Fix $x \in X$ and let $\{x_n\}$ be any sequence converging to x. Construct a sequence $\{y_n\}$ such that $y_n \in G(x_n)$ for all n. Since Γ is u.h.c., there exists a subsequence of $\{y_n\}$ whose limit point y is in $\Gamma(x)$. Call this subsequence $\{y_{n_k}\}$.

Let $z \in \Gamma(x)$. Since Γ is l.h.c., there exists $N \geq 1$ and a sequence $\{z_n\}_{n=N}^{\infty}$such that

$$z_n \to z \qquad \text{and} \qquad z_n \in \Gamma(x_n),\ \forall n \geq N.$$

Therefore every subsequence of $\{z_n\}$ converges to z. Take the subsequence $\{z_{n_k}\} \to z \in \Gamma(x)$.

Since $\varphi(x_{n_k}, y_{n_k}) \geq \varphi(x_{n_k}, z_{n_k})$ for all n_k and φ is continuous, it follows that $\varphi(x, y) \geq \varphi(x, z)$. Since this holds for any $z \in \Gamma(x)$, it follows that $y \in G(x)$. Hence G is u.h.c.

For the final part of this proof, to show that h is continuous, I refer you to the proofs provided in Sundaram (1996) and Stokey, Lucas and Prescott (1989). □

What does the Theorem of the Maximum mean? Roughly speaking the theorem tells us that continuity in the primitives is inherited by the solutions to the problem, but not in its entirety: some degree of continuity is lost in the process of optimization.

The Theorem of the Maximum under Convexity

Finally, if we make stronger assumptions on the primitives, we can obtain stronger results for the solutions. For the following results, it is helpful to define the graph of a correspondence.

Definition 5. The *graph* of a correspondence $\Gamma : X \to Y$ is the set

$$A = \{(x, y) \in X \times Y | y \in \Gamma(x)\}.$$

We now reconsider the Theorem of the Maximum in the context of convex optimization problems. That is, we place convexity restrictions on the primitives, in addition to the continuity and compactness conditions required by the original theorem.

Theorem 2. *(*The Theorem of the Maximum under Convexity.*) Let $X \subseteq \mathbb{R}^N$ and let $Y \subseteq \mathbb{R}^M$.*

Let $\varphi : X \times Y \to \mathbb{R}$ be a continuous function and let $\Gamma : X \to Y$ be nonempty, compact-valued, and continuous.

Let $h : X \to \mathbb{R}$ be defined in (2) *and $G : X \to Y$ be defined in* (3).

(i) If $\varphi(x, \cdot)$ is concave in y for every x and Γ is convex-valued, then the correspondence $G : X \to Y$ is convex-valued and u.h.c.

(ii) If $\varphi(x, \cdot)$ is strictly concave in y for every x and Γ is convex-valued, then the correspondence $G : X \to Y$ is single-valued and u.h.c. By Exercise 2, it is a continuous function.

(iii) If $\varphi(x, y)$ is concave on $X \times Y$ and Γ has a convex graph, then $h : X \to \mathbb{R}$ is a concave function. If "concave" is replaced with "strictly concave" then h is strictly concave.

Proof. Part (i). Fix $x \in X$ and take $y_1, y_2 \in G(x)$. Thus

$$\varphi(x, y_1) = \varphi(x, y_2) = h(x).$$

Let

$$y' = \theta y_1 + (1 - \theta) y_2$$

for some $\theta \in (0, 1)$. Since $\Gamma(x)$ is convex, $y' \in \Gamma(x)$. Then

$$\varphi(x, y') = \varphi(x, \theta y_1 + (1 - \theta) y_2).$$

By concavity of $\varphi(x, \cdot)$ in y,

$$\varphi(x, \theta y_1 + (1 - \theta) y_2) \geq \theta \varphi(x, y_1) + (1 - \theta) \varphi(x, y_2) = \theta h(x) + (1 - \theta) h(x) = h(x).$$

Therefore

$$\varphi(x, y') = h(x)$$

and as a result $y' \in G(x)$.

Part (ii). Fix $x \in X$. If $\varphi(x, \cdot)$ is strictly concave in y and $\Gamma(x)$ is convex, then the set of maximizers

$$\arg\max\{\varphi(x, y) | y \in \Gamma(x)\}$$

contains a single-point (see Lecture Notes 5 on convex optimization). Therefore $G(x)$ is single-valued. By the Theorem of the Maximum, it is u.h.c. By Exercise 2, it is a continuous function.

Part (iii). Let $x_1, x_2 \in X$. Let

$$x' = \theta x_1 + (1-\theta)x_2$$

for some $\theta \in (0,1)$. Pick any $y_1 \in G(x_1)$ and $y_2 \in G(x_2)$. Let

$$y' = \theta y_1 + (1-\theta)y_2.$$

Next, since $y_1 \in \Gamma(x_1)$ and $y_2 \in \Gamma(x_2)$ and Γ has a convex graph, then we must have that

$$y' \in \Gamma(x'),$$

meaning that given x', the choice y' is feasible but not necessarily optimal.

An optimum at x' satisfies:

$$h(x') \geq \varphi(x', y') = \varphi(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2)$$

By concavity of φ,

$$\begin{aligned} \varphi(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2) &\geq \theta\varphi(x_1, y_1) + (1-\theta)\varphi(x_2, y_2) \\ &= \theta h(x_1) + (1-\theta)h(x_2) \end{aligned} \tag{4}$$

Therefore

$$h(x') \geq \theta h(x_1) + (1-\theta)h(x_2)$$

which establishes concavity of h. If φ is strictly concave, the inequality in (4) becomes strict, proving strict concavity of h. □

Therefore, analogous to the continuity results of the Theorem of the Maximum, the convexity structure of the primitives is also inherited by the solutions, but again, not in its entirety.

Exercise 4. Consider the example in Exercise 3, part (b). Now let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a *bounded*, continuous function. Define the correspondence $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ by $\Gamma(x) = [0, f(x)]$.

You have already shown that Γ is continuous. Now show that Γ is both compact-valued and convex-valued. (This exercise is perhaps trivial, but it is an important case for what follows in this class.)

References

Stokey, Nancy L., Robert E. Lucas, and Edward C. Prescott, *Recursive Methods in Economic Dynamics*, Harvard University Press, 1989.

Sundaram, Rangarajan K., *A First Course in Optimization Theory*, Cambridge University Press, 1996.