Complete Markets

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Thus far we have assumed throughout a representative household—a fairly standard assumption in many macro models. However, underlying the representative household is an implicit assumption that markets are complete.

In this lecture we examine the complete markets assumption and its implications. I follow the treatment in Chapter 8 of Ljungqvist and Sargent (2004).

1 The Environment

As always, the environment consists of preferences and technology. In terms of technology, we consider a simple endowment economy. Time is discrete $t = 0, 1, \ldots$. In each period there is a realization of a stochastic event,

$$s_t \in S$$

which we call the "state." We let the history or sequence of events leading up to and including time t be denoted by

$$s^t = (s_0, s_1, \ldots, s_t).$$

The unconditional probability of observing a particular history s^t is given by

$$\pi(s^t)$$

This is the unconditional probability of history s^t from the standpoint of time 0.[1] We assume that all trade occurs after observing s_0, which means that $\pi(s_0) = 1$. At each point in time we assume that the history s^t is publicly observable.

There are N agents indexed by $i \in I \equiv (1, \ldots, N)$. There is one consumption good. Each agent i has a stochastic endowment of the consumption good given by $y^i(s_t)$ that depends on the realization of the state s_t. The household has preferences defined over a complete history-dependent consumption plan,

$$c^i = \{c^i(s^t)\}_{t,s^t}.$$

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[1] We will write conditional probabilities as

$$\pi\left(s^t | s^\tau\right)$$

which is the probability of observing s^t conditional on the realization of s^τ.

Preferences are given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) u(c^i(s^t))$$

where $\mathbb{E}_0$ is the expectation conditional on s_0. The period utility function u satisfies the usual regularity conditions: it is increasing, strictly concave, and satisfies the Inada conditions, $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$. Let us also assume it is twice differentiable to make our lives easier.

There is no storage in this economy, the consumption good is perishable: this means that the entire endowment in state s_t must be consumed today by the agents or thrown out.

Definition 1. An *allocation* in this economy is a complete history-dependent consumption plan for each agent:

$$\{c^i\}_{i\in I}.$$

Definition 2. An allocation is *feasible* if it satisfies

$$\sum_i c^i(s^t) \leq \sum_i y^i(s_t) \qquad \forall t, s^t. \tag{1}$$

This concludes our description of the environment.

Remark: History Dependence. One question we will be concerned with answering is whether the household's consumption exhibits history dependence. We say

Definition 3. A household's consumption at history s^t is **history-independent** if depends only on the current state s_t. Otherwise, we say that the household's consumption at history s^t is **history-dependent**.

Consumption at s^t is history-dependent if it depends not only on s_t but also on past states.

Note that each household's endowment at time t depends only on the current realized state, s_t. However, it would perhaps seem natural in equilibrium that the household's consumption at time t is history dependent—that is, household i's consumption may depend on past shocks and endowments. For example, you might think that if a household had a lucky streak of good shocks in the past, it would have higher wealth today, which means that its consumption would be high today (despite whatever is today's current endowment).

2 The Planner's Problem

Now that we have set up the environment we can first solve the planner's problem; this will give us the set of efficient (Pareto optimal) allocations. We will then compare the equilibrium allocation to the planner's allocation.

We thus consider a planner who attaches constant Pareto weights $\lambda_i > 0$ to each of the consumers and then chooses allocations in order to maximize welfare. We define the set of Pareto optimal allocations as follows.

Definition 4. An allocation $\{c^i\}_{i \in I}$ is Pareto efficient if it maximizes welfare

$$\sum_{i \in I} \lambda_i \left[\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) u(c^i(s^t)) \right]$$

subject to feasibility constraints

$$\sum_{i \in I} c^i(s^t) = \sum_{i \in I} y^i(s_t) \qquad \forall t, s^t.$$

for some set of Pareto weights $\lambda \equiv (\lambda_1, \ldots \lambda_N)$.

Note that there is a feasibility constraint for every date and every history. Let $\beta^t \pi(s^t) \theta(s^t)$ be the Lagrange multiplier on the feasibility constraint for time t and history s^t. We thus obtain the Lagrangian

$$L = \sum_{t=0}^{\infty} \sum_{s^t} \left\{ \sum_i \lambda_i \beta^t \pi(s^t) u(c^i(s^t)) - \beta^t \pi(s^t) \theta(s^t) \left(\sum_{i \in I} c^i(s^t) - \sum_{i \in I} y^i(s_t) \right) \right\}$$

Taking FOCs wrt to $c^i\left(s^t\right)$ we get

$$\lambda_i \beta^t \pi(s^t) u'(c^i(s^t)) - \beta^t \pi(s^t) \theta(s^t) = 0 \qquad \forall i, t, s^t$$

Therefore

$$\lambda_i u'(c^i(s^t)) = \theta(s^t) \qquad \forall i, t, s^t$$

Taking the ratio of this condition for two agents i and j, we get

$$\frac{u'(c^i(s^t))}{u'(c^j(s^t))} = \frac{\lambda_j}{\lambda_i}, \qquad \forall i, j, t, s^t. \tag{2}$$

That is, the planner sets the marginal rate of substitution of consumption between the two agents (or the ratio of marginal utilities) equal to the ratio of their Pareto weights. Therefore, note that in the Pareto optimal allocation, the ratios of marginal utilities between pairs of agents are constant across all histories and dates.

Solving (2) for $c^j(s^t)$ gives us

$$c^j(s^t) = u'^{-1} \left[\frac{\lambda_i}{\lambda_j} u'(c^i(s^t)) \right]$$

Finally, we can plug this into the feasibility condition for s^t. This gives us

$$\sum_j u'^{-1} \left[\frac{\lambda_i}{\lambda_j} u'(c^i(s^t)) \right] \leq Y(s_t) \tag{3}$$

where $Y(s_t) = \sum_i y^i(s_t)$ is the aggregate endowment at t, s^t.

This is the equation that determines $c^i(s^t)$. It only depends on the Pareto weights (which are constants) and the aggregate endowment at time t, state s_t. Since the right hand side is not history dependent, it must be the case that the left hand side is not history dependent either! Therefore, consumption of agent i, $c^i(s^t)$, depends *only* on the aggregate endowment in the current state s_t and not on the individual's entire history of income realizations. This is true for the consumption of all agents.

Furthermore, note that equation (3) implicitly defines $c^i(s^t)$ as a function of $Y(s_t)$, the aggregate endowment at t, s^t. We thus obtain the following proposition.

Proposition 1. *An efficient allocation is history-independent. Furthermore, in any efficient allocation, $c^i(s^t)$ is a time-invariant function of the aggregate endowment $Y(s_t)$.*

Thus, in any Pareto optimal allocation, we have that each household's consumption does not exhibit any history dependence. Furthermore, the consumption of each household does not even depend on its own endowment at s^t it is simply a function of the aggregate endowment! Therefore, consumption of each agent is perfectly correlated with the aggregate endowment or aggregate consumption.

CRRA Example. Suppose utility is homothetic:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

so that $u'(c) = c^{-\gamma}$. Then Pareto optimality (2) implies that

$$\left(\frac{c^i(s^t)}{c^j(s^t)}\right)^{-\gamma} = \frac{\lambda_j}{\lambda_i}$$

Solving this for $c^j(s^t)$ we get

$$c^j(s^t) = \left(\frac{\lambda_j}{\lambda_i}\right)^{1/\gamma} c^i(s^t)$$

Plugging this into the resource constraint, we have

$$\sum_j \left(\frac{\lambda_j}{\lambda_i}\right)^{1/\gamma} c^i(s^t) = Y(s_t)$$

Therefore, we may solve for agent i's consumption. It is given by

$$c^i(s^t) = \phi^i Y(s_t)$$

where ϕ^i is a time and state-invariant constant given by.

$$\phi^i = \frac{\lambda_i^{1/\gamma}}{\sum_j \lambda_j^{1/\gamma}}$$

Note that $\sum_i \phi^i = 1$. Thus, with CRRA (homothetic) utility each agent simply consumes a constant fraction of aggregate endowment (output). Agents face only aggregate risk in consumption; they do not face any idiosyncratic risk.

3 Complete Arrow-Debreu Markets

We now look at how the Pareto optimal allocation can be attained as a competitive equilibrium in a market with Arrow-Debreu securities.

Trading arrangements.

One could consider different trading arrangements and the competitive equilibrium that arises under these different structures. Consider two possible trading arrangements. The first trading arrangement will be one in which a market opens at time 0 and agents trade contingent claims to consumption for all times and all possible histories. After time 0, no further trades occur. We call this an Arrow-Debreu market structure: the dated contingent claims are called Arrow-Debreu securities; see Arrow and Debreu (1954); McKenzie (1954, 1959).

Another possible trading arrangement is one with sequential trade. In this market, trading occurs in every period, and agents trade only one-period ahead state contingent claims to consumption. This is called an Arrow market structure: the one-period ahead claims are called Arrow securities.

Both trading structures are complete market economies. In this lecture I will consider only the former: the Arrow-Debreu market. However, the two trading arrangements (sequential trade and Arrow-Debreu dated contingent-claims) are in fact equivalent; see Ljungqvist and Sargent (2004).

Arrow-Debreu Contingent Claims Market.

The Arrow-Debreu market works as follows. A market opens at time 0 in which all households may trade dated history-contingent claims to consumption. There exists a complete set of these securities, which we call Arrow-Debreu securities.

Specifically, a household can buy (or sell) a claim to one unit of consumption at time t, contingent on history s^t. We denote the price of this claim as $q(s^t)$: it is the price at time 0 of one unit of consumption at time t, history s^t. We call these prices Arrow-Debreu prices. The household's budget constraint at time 0 is thus given by

$$\sum_{t=0}^{\infty} \sum_{s^t} q(s^t) c^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q(s^t) y^i(s_t)$$

Note that this is a single budget constraint! All trade in this contingent claims market occurs at time 0. After time 0, trades that were agreed to at time 0 are executed, but no more trades occur.

We may thus write the household's problem as follows. The household chooses consumption to maximize its expected utility

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t)u(c^i(s^t))$$

subject to its budget constraint

$$\sum_{t=0}^{\infty}\sum_{s^t}q(s^t)c^i(s^t) \leq \sum_{t=0}^{\infty}\sum_{s^t}q(s^t)y^i(s_t)$$

We can normalize the price of consumption at time 0 to 1:

$$q_0(s_0) = 1$$

Therefore, the price system is in units of the time-0 consumption good.

Definition 5. A competitive equilibrium is an allocation

$$\{c^i\}_{i\in I}.$$

and a price system

$$\{q(s^t)\}_{t,s^t}$$

such that:

(i) given the price system, the allocation solves each household's problem, and

(ii) prices clear all markets.

$$\sum_i q(s^t)c^i(s^t) \leq \sum_i q(s^t)y^i(s_t) \qquad \forall t, s^t.$$

Note that market clearing implies that the economy's feasibility conditions (1) hold.

For household i's single budget constraint, let us attach μ_i as the Lagrange multiplier. We thus obtain the following first order conditions with respect to $c^i(s^t)$:

$$\beta^t\pi(s^t)u'\left(c^i(s^t)\right) - \mu_i q(s^t) = 0, \qquad \forall t, s^t \tag{4}$$

Taking the ratio of equation (4) for agents i and j implies

$$\frac{u'\left(c^i(s^t)\right)}{u'\left(c^j(s^t)\right)} = \frac{\mu_i}{\mu_j}$$

Thus, in the competitive equilibrium as in the pareto optimal allocation, the ratios of marginal utilities between pairs of agents are constant across all histories and dates. Again, solving this for $c^j(s^t)$ gives us

$$c^j(s^t) = u'^{-1}\left[\frac{\mu_j}{\mu_i}u'(c^i(s^t))\right]$$

Substituting this into our market clearing condition for t, s^t gives us

$$\sum_j u'^{-1}\left[\frac{\mu_j}{\mu_i}u'(c^i(s^t))\right] \leq Y(s_t)$$

Again, this is the equation that determines $c^i(s^t)$ in the competitive equilibrium. The following proposition is then immediate.

Proposition 2. *The competitive equilibrium allocation is not history dependent. Furthermore, it is a time-invariant function of the aggregate endowment $Y(s_t)$.*

Pareto optimality of the equilibrium allocation. Finally, we can compare the competitive equilibrium allocation to the set of pareto optimal allocations. The next proposition should be immediately obvious.

Proposition 3. *A competitive equilibrium allocation is a particular Pareto efficient allocation that sets the Pareto weights $\lambda_i = 1/\mu_i$ for all i, where μ_i is the unique (up to multiplication by a positive scalar) set of Lagrange multipliers (shadow values of wealth) associated with the competitive equilibrium.*

Furthermore at the competitive equilibrium allocation, the shadow prices $\beta^t\pi(s^t)\theta(s^t)$ for the associated planning problem are equal to the Arrow-Debreu prices $q(s^t)$ associated with the competitive equilibria.

This result should come as no surprise: the fact that the allocations for the planning problem and the competitive equilibrium are aligned reflects the two fundamental theorems of welfare.

CRRA Example. Again suppose that agents have homothetic utility. Then in the competitive equilibrium (as in the pareto optimal allocation)

$$c^j(s^t) = c^i(s^t)\left(\frac{\mu_i}{\mu_j}\right)^{\frac{1}{\gamma}}$$

Therefore, we will again have that

$$\sum_j \left(\frac{\mu_i}{\mu_j}\right)^{1/\gamma} c^i(s^t) = Y(s_t)$$

so that consumption of each agent is again a constant fraction of the aggregate endowment,

$$c^i(s^t) = \phi^i Y(s_t)$$

and this fraction ϕ^i assigned to each individual is independent of both time and state. Individual consumption is then perfectly correlated with aggregate consumption. Again agents face only aggregate risk in consumption–they do not face any idiosyncratic risk.

No Aggregate Shocks Example. Let s_t be drawn independently each period from the $U[0,1]$ distribution. Suppose there are two households, whose endowment processes are given by

$$y^1(s_t) = s_t \qquad \text{and} \qquad y^2(s_t) = 1 - s_t$$

In this case, the aggregate endowment $Y(s_t) = y^1(s_t)+y^2(s_t) = 1$ is constant across all states. In this economy then, the Pareto optimal allocation will have both $c^1(s^t)$ and $c^2(s^t)$ are constant over time (despite each household having random endowments). That is, there is complete risk sharing.

References

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