Dynamic Programming

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We can now draw on everything we've learned so far in order to study dynamic programming problems. These lecture notes draw on material found in Chapter 4 of Stokey, Lucas and Prescott (1989).

1 The Problem

Suppose time is discrete: $t = 0, 1, \ldots$ We are interested in problems of the following form.

1.1 The Sequence Problem (SP)

Choose the infinite sequence

$$\{x_{t+1}\}_{t=0}^{\infty},$$

in order to maximize the following objective:

$$\sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \varphi(x_t, x_{t+1}) \tag{1}$$

subject to

$$x_{t+1} \in \Gamma(x_t), \qquad \forall t = 0, 1, \ldots$$

and

$$x_0 \in X \quad \text{given.}$$

We call this the "sequence problem" because we are choosing infinite sequences, $\{x\}$.

In this problem, we let $X \subseteq \mathbb{R}^N$ be the set of all possible values for x. We let Γ denote the correspondence describing feasibility:

$$\Gamma : X \to X.$$

That is, for each $x_t \in X$, the set $\Gamma(x_t)$ describes the set of *feasible values* for x_{t+1} given the current state x_t.

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The function $\varphi(x_t, x_{t+1})$ is the current period return function and $\beta \in (0, 1)$ is the discount factor. Thus, the "givens" or "primitives" for this problem are the following:

$$\{X, \Gamma, \varphi, \beta\}.$$

We call a sequence $\{x_t\}_{t=0}^{\infty}$ a *plan*. Given $x_0 \in X$, let

$$\Pi(x_0) \equiv \{\{x_t\}_{t=0}^{\infty} | \quad x_{t+1} \in \Gamma(x_t), \quad \forall t\}$$

denote the set of all plans (sequences) that are *feasible* given initial condition x_0.

Example: The Neoclassical Growth Model

In Lecture Notes 3 we considered the planning problem in the Neoclassical Growth Model. Let us return to that example.

Planner's Problem. Given an initial level of capital, $k_0 > 0$, the social planner chooses an infinite sequence for consumption and capital,

$$\{c_t, k_{t+1}\}_{t=0}^{\infty},$$

so as to maximize the utility of the representative household:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t) \tag{2}$$

with $\beta \in (0, 1)$, subject to the resource constraint,

$$c_t + k_{t+1} = f(k_t) + (1 - \delta)k_t, \qquad \forall t \geq 0 \tag{3}$$

and non-negativity constraints,

$$c_t \geq 0, \qquad k_{t+1} \geq 0, \qquad \forall t \geq 0.$$

This is an example of the sequence problem. Let

$$\gamma(k_t) \equiv f(k_t) + (1 - \delta)k_t \tag{4}$$

denote the total goods available. Then

$$c_t = \gamma(k_t) - k_{t+1}.$$

We can thus write the problem as

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(\gamma(k_t) - k_{t+1})$$

subject to

$$0 \leq k_{t+1} \leq \gamma(k_t)$$

with $k_0 > 0$ given.

In this case the current period return function is given by

$$\varphi(k_t, k_{t+1}) = U(\gamma(k_t) - k_{t+1})$$

and the feasibility correspondence Γ is defined by

$$\Gamma(k_t) = [0, \gamma(k_t)].$$

1.2 The Functional Equation (FE)

Corresponding to any such sequence problem, we have the functional equation (FE) of the form:

$$v(x) = \sup_{y \in \Gamma(x)} \left[\varphi(x, y) + \beta v(y) \right], \qquad \text{for all } x \in X. \tag{5}$$

We call the function $v : X \to \mathbb{R}$ defined in (5) the *value function*. In this problem, we call x the beginning-of-period *state variable*, and y the *control* or *choice* variable. In particular, y is the variable that is *chosen* today and that will become the state variable tomorrow. Finally, $\varphi(x, y)$ is the current period return given state variable x and control y.

2 The Principal of Optimality

We seek to establish a relationship between the solutions to these two problems and develop methods for analyzing them. The general idea is that the solution to the sequence problem in (1) is a solution to the functional equation in (5).

Towards this goal, we start by making the following set of relatively weak assumptions on the primitives $(X, \Gamma, \varphi, \beta)$.

Assumption 1. (i) $X \subseteq \mathbb{R}^N$.
(ii) $\Gamma(x)$ is nonempty for all $x \in X$.
(iii) $\varphi : X \times X \to \mathbb{R}$ is bounded.
(iv) $\beta \in (0, 1)$.

Part (ii) of these assumptions ensures that the set of feasible plans $\Pi(x_0)$ is nonempty for each $x_0 \in X$.

One can relax part (iii) and (iv) a bit for the following theorems; see Stokey, Lucas and Prescott (1989) for details. In fact, for what follows, parts (iii) and (iv) can simply be replaced with $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t \varphi(x_t, x_{t+1})$ exists. φ bounded and $\beta \in (0, 1)$ are sufficient for this limit to exist; I simply impose this. Hence the objective function in the SP is well defined for every plan

$\{x\} \in \Pi(x_0)$. In particular, let us define $u : \Pi(x_0) \to \mathbb{R}$ by

$$u(\{x\}) = \lim_{n\to\infty} \sum_{t=0}^{n} \beta^t \varphi(x_t, x_{t+1}) = \sum_{t=0}^{\infty} \beta^t \varphi(x_t, x_{t+1})$$

That is, u is the infinite sum of discounted returns from the feasible sequence $\{x\} \in \Pi(x_0)$. To see that φ bounded and $\beta \in (0, 1)$ are sufficient for this limit to exist, note that if B is a bound for $|\varphi(x_t, x_{t+1})|$, then

$$|u(\{x\})| \leq \frac{B}{1-\beta}.$$

We can define the *supremum function* $v^* : X \to \mathbb{R}$ by

$$v^*(x_0) = \sup_{\{x\}\in\Pi(x_0)} u(\{x\}). \tag{6}$$

That is, $v^*(x_0)$ is the supremal value of the objective function of the sequence problem (1). It follows by definition of v^* that v^* is the unique function that satisfies:

(i) for all $\{x\} \in \Pi(x_0)$,

$$v^*(x_0) \geq u(\{x\}), \tag{7}$$

and (ii) for any $\epsilon > 0$,

$$v^*(x_0) \leq u(\{x\}) + \epsilon \tag{8}$$

for some $\{x\} \in \Pi(x_0)$. [Note: this should look familiar from Lecture Notes 4 on optimization.]

Our interest is in the supremum function defined in (6), i.e. the supremal value in the sequence problem, and its connection to the solutions to the functional equation in (5). In interpreting these results, keep in mind that the supremum function v^* is always uniquely defined by (6). On the other hand, the functional equation may have zero, one, or many solutions—this issue will be dealt with later.

In order to establish this connection, we will say that the supremum function v^* satisfies the functional equation in (5) if two conditions hold:

(i) for all $y \in \Gamma(x_0)$,

$$v^*(x_0) \geq \varphi(x_0, y) + \beta v^*(y) \tag{9}$$

and (ii) for any $\epsilon > 0$,

$$v^*(x_0) \leq \varphi(x_0, y) + \beta v^*(y) + \epsilon \tag{10}$$

for some $y \in \Gamma(x_0)$.

Before we prove that the supremum function satisfies the functional equatoin, it is useful to establish a preliminary result.

Lemma 1. *For any $x_0 \in X$ and any sequence $\{x_0, x_1, \ldots\} = \{x\} \in \Pi(x_0)$,*

$$u(\{x\}) = \varphi(x_0, x_1) + \beta u(\{x'\})$$

where $\{x'\} = \{x_1, x_2, \ldots\}$.

Proof. For any $x_0 \in X$ and any $\{x\} \in \Pi(x_0)$,

$$\begin{aligned} u(\{x\}) &= \sum_{t=0}^{\infty} \beta^t \varphi(x_t, x_{t+1}) = \varphi(x_0, x_1) + \sum_{t=1}^{\infty} \beta^t \varphi(x_t, x_{t+1}) \\ &= \varphi(x_0, x_1) + \beta \sum_{t=0}^{\infty} \beta^t \varphi(x_{t+1}, x_{t+2}) \\ &= \varphi(x_0, x_1) + \beta u(\{x'\}) \end{aligned}$$

as was to be shown. □

Theorem 1. *Let X, Γ, φ, and β satisfy Assumption 1. Then the supremum function v^* satisfies the functional equation* (5).

Proof. Fix $x_0 \in X$. We have that v^* is the unique function that satisfies (7) and (8). It suffices to show that v^* satisfies 9 and 10.

To establish 9, let $x_1 \in \Gamma(x_0)$ and $\epsilon > 0$ given. Then by 8, there exists $\{x'\} = \{x_1, x_2, \ldots\} \in \Pi(x_1)$ such that

$$u(\{x'\}) \geq v^*(x_1) - \epsilon.$$

Note further that $\{x\} = \{x_0, x_1, x_2, \ldots\} \in \Pi(x_0)$ by the fact that $x_1 \in \Gamma(x_0)$. From 7:

$$v^*(x_0) \geq u(\{x\})$$

and from Lemma 1,

$$u(\{x\}) = \varphi(x_0, x_1) + \beta u(\{x'\})$$

Thus

$$\begin{aligned} v^*(x_0) &\geq \varphi(x_0, x_1) + \beta u(\{x'\}) \\ &\geq \varphi(x_0, x_1) + \beta v^*(x_1) - \beta\epsilon \end{aligned}$$

Since ϵ was arbitrary, it follows that v^* satisfies 9.

Next, to establish 10, again let $\epsilon > 0$. From 8 one can choose a sequence $\{x\} \in \Pi(x_0)$ such that

$$v^*(x_0) \leq u(\{x\}) + \epsilon = \varphi(x_0, x_1) + \beta u(\{x'\}) + \epsilon$$

where $\{x'\} = \{x_1, x_2, \ldots\}$ and the second part follows from Lemma 1. It follows from 7 that

$$v^*(x_0) \leq \varphi(x_0, x_1) + \beta v^*(x_1) + \epsilon.$$

Since $x_1 \in \Gamma(x_0)$, it follows that v^* satisfies 10. □

The next theorem provides a partial converse to Theorem 1. For this proof, we define, for each $n = 0, 1, \ldots$, the function $u_n : \Pi(x_0) \to \mathbb{R}$ by

$$u_n(\{x\}) = \sum_{t=0}^{n} \beta^t \varphi(x_t, x_{t+1}).$$

That is, u_n is the partial sum of discounted returns in periods 0 to n from the feasible sequence $\{x\}$.

Theorem 2. *Let X, Γ, φ, and β satisfy Assumption 1. If v is a solution to the functional equation* (5) *and satisfies:*

$$\lim_{n\to\infty} \beta^n v(x_n) = 0, \qquad \forall \{x\} \in \Pi(x_0), x_0 \in X, \tag{11}$$

then $v = v^*$.

Proof. Fix $x_0 \in X$. We have that v is a function that satisfies (9) and (10). It suffices to show that v satisfies (7) and (8).

Note that (9) implies that for all $\{x\} \in \Pi(x_0)$,

$$\begin{aligned} v(x_0) \geq & \varphi(x_0, x_1) + \beta v(x_1) \\ \geq & \varphi(x_0, x_1) + \beta \varphi(x_1, x_2) + \beta^2 v(x_2) \\ & \vdots \\ \geq & u_n(\{x\}) + \beta^{n+1} v(x_{n+1}) \end{aligned}$$

for all $n = 1, 2, \ldots$. Taking the limit as $n \to \infty$ and using condition (11), it follows that v satisfies (7).

Next, to establish that v satisfies (8), let $\epsilon > 0$. Choose a strictly-positive sequence $\{\delta_t\}_{t=1}^{\infty}$ in $\mathbb{R}$ such that $\delta_t > 0$ for all t and:

$$\sum_{t=1}^{\infty} \beta^{t-1} \delta_t \leq \epsilon.$$

For example, one can choose the following sequence: $\delta_t = \bar{\delta} > 0$ is a constant for all t. Then

$$\sum_{t=1}^{\infty} \beta^{t-1} \bar{\delta} = \bar{\delta} \sum_{t=0}^{\infty} \beta^t = \frac{1}{1-\beta} \bar{\delta}$$

Therefore one can just choose $\bar{\delta} \leq (1-\beta)\epsilon$.

Since (10) holds (and replacing ϵ with δ), we can choose a sequence:

$$x_1 \in \Gamma(x_0), x_2 \in \Gamma(x_1), \ldots$$

such that

$$v(x_t) \leq \varphi(x_t, x_{t+1}) + \beta v(x_{t+1}) + \delta_{t+1}, \qquad \forall t = 0, 1, \ldots$$

By construction, $\{x\} = \{x_0, x_1, x_2, \ldots\} \in \Pi(x_0)$ and

$$
\begin{aligned}
v(x_0) \leq & \sum_{t=0}^{n} \beta^t \varphi(x_t, x_{t+1}) + \beta^{n+1} v(x_{n+1}) + (\delta_1 + \beta\delta_2 + \cdots + \beta^n \delta_{n+1}) \\
\leq & u_n(\{x\}) + \beta^{n+1} v(x_{n+1}) + \epsilon,
\end{aligned}
$$

for all $n = 1, 2, \ldots$. Using condition (11), for all n sufficiently large,

$$v(x_0) \leq u_n(\{x\}) + \epsilon.$$

Since ϵ was arbitrary, it follows that v satisfies 8. □

Theorem 1 tells us that v^*, i.e. the solution to the sequence problem, satisfies the functional equation (FE). Conversely, Theorem 2 tells us that if a function v satisfies both the functional equation and a certain boundary condition, then it also solves the sequence problem.

It is an immediate consequence of this theorm that the functional equation has at most one solution that satisfies the boundary condition in (11). The functional equation may have other solutions as well, but Theorem 2 shows that these other solutions always violate (11).

The next task is to characterize the set of feasible plans that attain the optimum. We call a feasible plan

$$\{x^*\} \in \Pi(x_0)$$

an *optimal plan* from x_0 if it attains the supremum in the SP. That is, $\{x^*\}$ is an optimal plan if

$$\sum_{t=0}^{\infty} \beta^t \varphi(x_t^*, x_{t+1}^*) = v^*(x_0)$$

The next theorems provide a relationship between optimal plans and plans that satisfy the functional equation.

Theorem 3. *Let X, Γ, φ, and β satisfy Assumption 1. Let $\{x^*\} \in \Pi(x_0)$ be an optimal plan, i.e. it attains the supremum in the sequence problem given initial state x_0. Then*

$$v^*(x_t^*) = \varphi(x_t^*, x_{t+1}^*) + \beta v^*(x_{t+1}^*), \qquad \forall t = 0, 1, \ldots.$$

Proof. Omitted; see Stokey, Lucas and Prescott (1989). □

The next theorem provides a partial converse to Theorem 3. For this, we need to first define the limit superior (limsup) of a sequence.

Definition 1. The *limit superior* of a sequence $\{x_n\}$ is defined as

$$\lim_{n\to\infty} \sup x_n = \lim_{n\to\infty} \left(\sup_{m \geq n} x_n \right).$$

One can think of the limit superior as the limiting least upper bound of a sequence.

Theorem 4. *Let X, Γ, φ, and β satisfy Assumption 1.*

Let $\{x^*\} \in \Pi(x_0)$ be a feasible plan from x_0 that satisfies

$$v^*(x_t^*) = \varphi(x_t^*, x_{t+1}^*) + \beta v^*(x_{t+1}^*), \qquad \forall t = 0, 1, \ldots. \tag{12}$$

and

$$\lim_{t\to\infty} \sup \beta^t v^*(x_t^*) \leq 0, \tag{13}$$

Then $\{x^*\}$ attains the supremum in SP given initial state x_0.

Proof. Suppose that $\{x^*\} \in \Pi(x_0)$ satisfies (12) and (13). It follows by iterating on(12) that

$$\begin{aligned} v^*(x_0) &= \varphi(x_0, x_1^*) + \beta v^*(x_1^*) \\ &= \varphi(x_0, x_1^*) + \beta\varphi(x_1^*, x_2^*) + \beta^2 v^*(x_2^*) \\ &\vdots \\ &= u_n(\{x^*\}) + \beta^{n+1} v^*(x_{n+1}^*) \end{aligned}$$

for all $n = 1, 2, \ldots$. Taking the limit as $n \to \infty$ and using (13), it follows that

$$v^*(x_0) \leq u(\{x^*\}).$$

Since $\{x^*\} \in \Pi(x_0)$, the reverse inequality holds:

$$v^*(x_0) \geq u(\{x^*\}).$$

Therefore $v^*(x_0) = u(\{x^*\})$. □

Thus, while Theorems 1 and 2 give us an equivalence between the supremum function and the value function, Theorems 3 and 4 give us an equivalence between an optimal plan and a plan that satisfies the functional equation with $v = v^*$.

Next we define a policy correspondence.

Definition 2. A *policy correspondence* is a nonempty correspondence $G : X \to X$ with $G(x) \subseteq \Gamma(x)$ for all $x \in X$. If G is single-valued, we call it a policy function and denote it by a lower case g. Given x_0, we say that a sequence $\{x\} \in \Pi(x_0)$ is *generated by* G if it satisfies

$$x_{t+1} \in G(x_t), \qquad \forall t = 0, 1, \ldots$$

We next define the optimal policy correspondence.

Definition 3. Let v^* be the supremum function (which we have shown satisfies the functional equation (5)). Then the *optimal policy correspondence* G^* is the policy correspondence defined by

$$G^*(x) = \{y \in \Gamma(x) | v^*(x) = \varphi(x, y) + \beta v^*(y)\}.$$

If G^* is single-valued, we call it the *optimal policy function* and denote it by a lower case g^*.

In other words, for any $x \in X$, the set $G^*(x)$ is the set of y values that attains the supremum in (5). Using this definition of the optimal policy correspondence (or function, if single-valued) we can rewrite Theorems 3 and 4 in the following way.

Theorem 5. *Let X, Γ, φ, and β satisfy Assumption 1.*
(i) Let $\{x^*\} \in \Pi(x_0)$ be an optimal plan. Then $\{x^*\}$ is generated by G^*.
(ii) Any plan generated by G^* and satisfying (13) is an optimal plan.

Proof. Follows from Theorems 3 and 4. □

We have thus established the tight connection between the solutions of the two problems—the sequence problem and the functional equation. Richard Bellman called this connection the "Principal of Optimality."

3 Uniqueness of the Value Function

As we have mentioned above, the functional equation may have multiple solutions. We now show that under certain conditions it has a unique solution. In order to do so, we make some stronger assumptions on the primitives of problem.

Assumption 2. (i) $X \subseteq \mathbb{R}^N$ is convex.
(ii) $\Gamma : X \to X$ is nonempty, compact-valued, and continuous.
(iii) $\varphi : X \times X \to \mathbb{R}$ is bounded and continuous.
(iv) $\beta \in (0, 1)$.

Note that this subsumes the assumptions made in Assumption 1. Next, recall from Lecture Notes 1 & 2 we defined $C(X)$ to be the set of bounded, continuous functions

$$f : X \to \mathbb{R}$$

with the sup norm

$$\|f\| = \sup_{x \in X} |f(x)|.$$

Furthermore, recall that at the end of Lecture Notes 1, we proved that $C(X)$ with the sup norm is a complete, normed vector space (a Banach space).

We next define the operator T on the space $C(X)$ as follows:

$$Tf(x) = \max_{y \in \Gamma(x)} [\varphi(x, y) + \beta f(y)] \tag{14}$$

for all functions $f \in C(X)$. Given this operator, our functional equation in (5) may then be written as follows:

$$v = Tv.$$

That is, we have defined the operator T so that the value function v defined in (5) is a fixed point of T.

We will now use the Contraction Mapping Theorem, Blackwell's sufficiency conditions, the Weierstrass Theorem, and the Theorem of the Maximum, to prove uniqueness of the value function defined in (5).

Theorem 6. *Let X, Γ, φ, and β satisfy Assumption 2. Then the operator T defined in* (14) *maps $C(X)$ into itself,*

$$T : C(X) \to C(X)$$

and T has a unique fixed point $v \in C(X)$,

$$Tv = v,$$

and for any $f_0 \in C(X)$,

$$||T^n f_0 - v|| \leq \beta^n ||f_0 - v||, \qquad \forall n = 0, 1, 2, \ldots$$

Moreover, given the fixed point v, the policy correspondence G defined by

$$G(x) = \{y \in \Gamma(x) | v(x) = \varphi(x, y) + \beta v(y)\}. \tag{15}$$

is non-empty, compact-valued, and u.h.c.

Proof. We want to show that T defined in (14) is a contraction on the function space $C(X)$.

We first need to show that the operator T does in fact map $C(X)$ into itself. Note that for each $x \in X$, the functions φ and f are both continuous and bounded by assumption and the set $\Gamma(x)$ is non-empty and compact. Therefore, for each $f \in C(X)$ and $x \in X$, the problem in (14) is that of maximizing a continuous function $\varphi(x, \cdot) + \beta f(\cdot)$ over a compact set $\Gamma(x)$. By the Weierstrass theorem, the maximum is attained. Since φ and f are both bounded, it follows that the function Tf is bounded.

Next, φ and f are both continuous functions and $\Gamma : X \to X$ is nonempty, compact-valued, and continuous correspondence. Invoking the Theorem of the Maximum it follows that Tf is also continuous. Therefore, Tf is both bounded and continuous,which implies:

$$T : C(X) \to C(X).$$

The next step is to show that the operator T is a contraction on $C(X)$. For this, it suffices to show that it satisfies Blackwell's sufficiency conditions. We consider both conditions.

(i) (*monotonicity*) Take two functions $f, h \in C(X)$ such that $f(x) \leq h(x)$ for all $x \in X$. Applying T to f:

$$Tf(x) = \max_{y \in \Gamma(x)} [\varphi(x, y) + \beta f(y)]$$

Fix an $x \in X$. Let $G(x; f)$ be the set of y values that attains the maximum given f, that is:

$$G(x; f) \equiv \{y \in \Gamma(x) | Tf(x) = \varphi(x, y) + \beta f(y)\}. \tag{16}$$

For any $y \in G(x; f) \subseteq \Gamma(x)$,

$$Tf(x) = \varphi(x, y) + \beta f(y) \le \varphi(x, y) + \beta h(y), \qquad \forall x \in X$$

since $f(y) \le h(y)$ for all $y \in \Gamma(x)$. Furthermore,

$$\varphi(x, y) + \beta h(y) \le \max_{y' \in \Gamma(x)} \left[\varphi(x, y') + \beta h(y')\right] = Th(x), \qquad \forall x \in X$$

since $y \in \Gamma(x)$ is feasible. Therefore:

$$Tf(x) \le Th(x), \qquad \forall x \in X.$$

and monotonicity is satisfied.

(ii) (*discounting*) Next for any function $f \in C(X)$,

$$T(f + a)(x) = \max_{y \in \Gamma(x)} [\varphi(x, y) + \beta(f(y) + a)] = \max_{y \in \Gamma(x)} [\varphi(x, y) + \beta f(y)] + \beta a$$

Therefore

$$T(f + a)(x) = Tf(x) + \beta a, \quad \forall f \in C(X).$$

and discounting is satisfied.

Therefore the operator T satisfies Blackwell's sufficiency conditions and is thereby a contraction on $C(X)$. Next because $C(X)$ with the sup norm is a complete, normed vector space, we can invoke the Contraction Mapping Theorem. That is: (i) T has exactly one fixed point $v \in C(X)$,

$$Tv = v, \text{ and}$$

(ii) for any $f_0 \in C(X)$,

$$||T^n f_0 - v|| \le \beta^n ||f_0 - v||, \qquad \forall n = 0, 1, 2, \ldots$$

Finally, with the fixed point v, the stated properties of the correspondence G defined in (15) follow from the Theorem of the Maximum. □

Theorem 6 proves existence and uniqueness of the fixed point to the functional equation. It follows immediately from Theorem 2 that the unique solution, i.e. the unique bounded continuous function satisfying the functional equation (5), is equal to the supremum function v^* for the associated sequence problem. [Note that (11) trivially holds for this solution.]

Another useful aspect of Theorem 6 is that it gives us guidance as to *how* one can obtain the unique solution. In macro when we start solving these problems numerically (on the computer), Theorem 6 effectively gives us an algorithm for finding the unique solution. This algorithm is to simply iterate on the Bellman equation according to (14). This numerical iteration *we know* is a contraction, and by Theorem 6 it converges in the sup norm to a unique fixed point.

Finally, there are more conditions and theorems that characterize the value function and the policy function more sharply. We consider these next.

4 Properties of the value function and the policy function

To characterize the value function and the policy function more sharply, we need more information about the return function φ and the feasibility correspondence Γ. In this section we impose more assumptions on the primitives relative to Assumption 1.

For the following theorems, to make things simpler, let $X \subseteq \mathbb{R}$ (i.e. we reduce the dimension of X to $N = 1$). We maintain the assumption that X is convex.

4.1 Monotonicity of the Value Function

We first add an assumption to ensure that the value function is strictly increasing.

Assumption 3. (i) For every y, $\varphi(\cdot, y)$ is strictly increasing in x, and (ii) Γ is monotone in the sense that $x \leq x'$ implies

$$\Gamma(x) \subseteq \Gamma(x')$$

Theorem 7. *Let $X, \Gamma, \varphi, \beta$ satisfy Assumptions 1-3, and let v be the unique solution to (5). Then v is strictly increasing.*

Proof. Omitted; see Stokey, Lucas and Prescott (1989). □

This is fairly intuitive: if φ is strictly increasing in x and everything that was feasible for a given x is still feasible for any $x' \geq x$, then the value function is strictly increasing in x. In other words, as x increases, things get better.

4.2 Concavity of the Value Function

We next add additional assumptions to ensure strict concavity of v. Let A denote the *graph* of $\Gamma : X \to Y$, given by

$$A = \{(x, y) \in X \times Y | y \in \Gamma(x)\}.$$

Assumption 4. $\varphi : X \times X \to \mathbb{R}$ is strictly concave and A, the graph of Γ, is convex.

By φ strictly concave, we mean that for all $\theta \in (0, 1)$, and all $(x, y), (x', y') \in A$,

$$\varphi[\theta(x, y) + (1 - \theta)(x', y')] \geq \theta\varphi(x, y) + (1 - \theta)\varphi(x', y')$$

and with strict inequality if $x \neq x'$.

Furthermore, by A convex, we mean that for any $\theta \in [0, 1]$, and $x, x' \in X$,

$$y \in \Gamma(x) \text{ and } y' \in \Gamma(x')$$

implies

$$\theta y + (1 - \theta)y' \in \Gamma(\theta x + (1 - \theta)x').$$

Theorem 8. *Let $X, \Gamma, \varphi, \beta$ satisfy Assumptions 2-4. Let v be the unique solution to* (5) *and G satisfy* (15)*. Then v is strictly concave and G is a continuous, single-valued function, which we denote by g.*

Proof. I will not prove that v is strictly concave; I will only prove that with v strictly concave, G is a continuous, single-valued function.

Fix $x \in X$ and consider the problem:

$$\max_{y \in \Gamma(x)} [\varphi(x, y) + \beta v(y)]$$

where the function v is bounded, continuous, and strictly concave. By Assumption 4, $\varphi(x, \cdot)$ is concave in y and the set $\Gamma(x)$ is convex. It follows that the maximum is attained at a unique y value. Hence $G(x)$ is single-valued. Continuity follows from the fact that it is upper hemi-continuous (see Lecture Notes 6, Theorem 2: Theorem of the Maximum under Convexity). □

Theorem 8 is similar in spirit to the Theorem of the Maximum under Convexity. As long as the current period return function is strictly concave, and the feasibility correspondence has a convex graph, then the value function v is strictly concave, and G is a continuous, single-valued function, which we denote by g. We call g the *policy function*.

The basic idea behind both Theorems 7 and 8 is that the operator T preserves certain properties of the primitives.

4.3 Application to the NGM

Consider again the planner's problem in the NGM

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(\gamma(k_t) - k_{t+1})$$

subject to

$$0 \leq k_{t+1} \leq \gamma(k_t)$$

with $k_0 > 0$ given.

For the purposes of this exercise, suppose there is full depreciation of capital: $\delta = 1$, so that $\gamma(k_t) = f(k_t)$.

Let $\mathbb{R}_+$ (the set of non-negative reals) be the set of all possible values for k. We can write the Bellman (functional) equation as:

$$v(k) = \max_{k' \in \Gamma(k)} \left[\varphi(k, k') + \beta v(k')\right] \tag{17}$$

where the current period return function is given by

$$\varphi(k, k') = U(f(k) - k')$$

and the feasibility correspondence Γ is defined by:

$$\Gamma(k) = [0, f(k)] \subseteq \mathbb{R}_+.$$

Exercise 1. In the Bellman equation in (17), what does k and k' represent? Which one is the state variable and which one is the control variable?

Next, we make the following assumptions on preferences and technology:

Assumption 5. (i) $\beta \in (0, 1)$.
(ii) $U : \mathbb{R}_+ \to \mathbb{R}$ is continuous, bounded, strictly increasing, and strictly concave.
(iii) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, bounded, strictly increasing, and strictly concave.

Exercise 2. Given Assumption 5 and using the theorems provided above, show that the value function v defined in (17) is:
(a) unique,
(b) strictly increasing,
(c) strictly concave and its associated policy correspondence is single-valued and continuous, i.e. a continuous function.

Note: for each of these, you simply need to verify that the sufficient conditions of the relevant theorem hold for this problem, and then just apply the theorem.

4.4 Differentiability of the value function

Finally, now that existence and uniqueness of a solution to the functional equation has been established, we would like to treat the maximation problem in that equation as an ordinary programming problem and use the standard methods of calculus to characterize the policy function g. But in order to know that we can use calculus, we need to know something about the differentiability of v. We do that next.

We start with a theorem by Benveniste and Scheinkman (1979). Under fairly general conditions, the value function v is differentiable.

Theorem 9. *(*Benveniste-Scheinkman.*) Let $X \subseteq \mathbb{R}$ be convex. Let $V : X \to \mathbb{R}$ be a concave function, let $x_0 \in$ int(X), and let $B_\epsilon(x_0)$ be an open ball centered at x_0. If there is a concave, differentiable function $W : B_\epsilon(x_0) \to \mathbb{R}$ with*

$$W(x_0) = V(x_0),$$

and with

$$W(x) \leq V(x), \qquad \forall x \in B(x_0)$$

then V is differentiable at x_0 with its derivative given by

$$V'(x_0) = W'(x_0).$$

Proof. Omitted; see Stokey, Lucas and Prescott (1989). □

I omit the proof but the idea behind this theorem is rather simple if you look at Figure 1. Applying this result to our problem is straightforward, given the following additional restriction.

Assumption 6. φ is continuously differentiable in x.

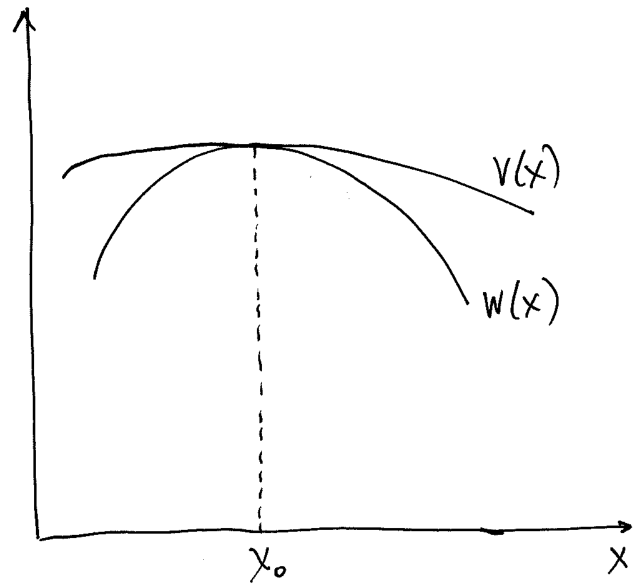

Figure 1. The Benveniste-Scheinkman Theorem

Theorem 10. *Let $X, \Gamma, \varphi, \beta$ satisfy Assumptions 2-4 and Assumption 6, and let v be the unique solution to* (5) *and g satisfy (15). If $x_0 \in$ int(X), and $g(x_0) \in$ int$(\Gamma(x_0))$, then v is continuously differentiable at x_0 with its derivative given by*

$$v'(x_0) = \left. \frac{\partial \varphi(x, g(x_0))}{\partial x} \right|_{x=x_0}. \tag{18}$$

Proof. Given $\epsilon > 0$, define a new function W on $B_\epsilon(x_0)$ by

$$W(x) \equiv \varphi(x, g(x_0)) + \beta v(g(x_0)), \qquad \forall x \in B_\epsilon(x_0).$$

Since φ is strictly concave and differentiable it follows that W is concave and differentiable. Moreover, for all $x \in B_\epsilon(x_0)$,

$$W(x) \leq \max_{y \in \Gamma(x)} [\varphi(x, y) + \beta v(y)] = v(x),$$

which holds with equality at x_0. We can then apply the Benveniste-Scheinkman Theorem and the result follows immediately:

$$v'(x_0) = W'(x_0) = \left. \frac{\partial \varphi(x, g(x_0))}{\partial x} \right|_{x=x_0}.$$

□

Equation (18) in Theorem 10 is essentially an envelope condition. Thus, we have established that under these conditions the value function is differentiable and we can use standard methods of calculus to characterize the value function. This will be useful in macro.

References

Stokey, Nancy L., Robert E. Lucas, and Edward C. Prescott, *Recursive Methods in Economic Dynamics*, Harvard University Press, 1989.