# Aggregation, Asset Pricing, and the Consumption CAPM

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In this lecture we apply the complete markets (Arrow-Debreu) paradigm to Asset Pricing. This lecture builds on material found in Chapters 8 and 13 of Ljungqvist and Sargent (2004).

## 1 Arrow-Debreu Securities and Arrow-Debreu Prices

First, recall the environment we considered in the previous lecture. We define the set of Arrow-Debreu securities: an AD security is a claim to one unit of consumption at time $t$, contingent on history $s^t$. We denote the price of this claim as $q(s^t)$: it is the price *at time 0* of one unit of consumption at time $t$, history $s^t$. Furthermore, we call these prices Arrow-Debreu prices.

The household's budget constraint at time 0 is given by

$$\sum_{t=0}^{\infty}\sum_{s^t} q(s^t)c^i(s^t) \leq \sum_{t=0}^{\infty}\sum_{s^t} q(s^t)y^i(s_t)$$

We assume that $q(s_0) = 1$ so that the consumption at time 0 good is the numeraire: all prices are in terms of this good.

Finally, recall that attaching $\mu_i$ as the Lagrange multiplier to the household's budget constraint, we obtain the following first order conditions with respect to $c^i\left(s^t\right)$:

$$\beta^t \pi(s^t)u'(c^i(s^t)) - \mu_i q(s^t) = 0 \qquad \text{for all } t, s^t \tag{1}$$

where $\pi(s^t)$ is the unconditional probability of history $s^t$. Taking the ratio of equation (1) for agents $i$ and $j$ implies

$$\frac{u'(c^i(s^t))}{u'(c^j(s^t))} = \frac{\mu_i}{\mu_j}$$

**Proposition 1.** *Let $\varphi_i \equiv 1/\mu_i$ be the set of Negishi weights (unique up to a scalar constant). A competitive equilibrium allocation is a particular Pareto efficient allocation that sets the Pareto weights equal to the Negishi weights, $\lambda_i = \varphi_i$ for all $i$.*

## 2 Aggregation and the Existence of a Representative Household

We now examine how the completeness of markets admits a representative household (consumer).

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Suppose there are two households, $1$ and $2$, with heterogeneous per-period utility functions $u:\mathbb{R}_+\to\mathbb{R}$ and $v:\mathbb{R}_+\to\mathbb{R}$,

$$u(c^i) \qquad \text{and} \qquad v(c^j).$$

We assume $u:\mathbb{R}_+\to\mathbb{R}$ and $v:\mathbb{R}_+\to\mathbb{R}$ are both strictly increasing and strictly concave.

Let $\varphi$ be the Negishi weight on household 1, and $1-\varphi$ be the Negishi weight on household 2. Consider the Pareto problem

$$U(C(s^t)) \equiv \max_{\{c^i(s^t),c^j(s^t)\}} \varphi u(c^i(s^t)) + (1-\varphi)v(c^j(s^t))$$

subject to the feasibility condition

$$c^i(s^t)+c^j(s^t)=C(s^t)=y^1(s^t)+y^2(s^t)$$

where $C(s^t)$ is aggregate consumption (equal to the aggregate endowment) in history $s^t$. Note that we can drop the $s^t$ from this problem: it is the same in every state and history; what matters is $C$.

Let us define functions $x^i(C)$ and $x^j(C)$ as the consumption allocations of households 1 and 2, respectively, that solve the planner's problem. That is,

$$\left\{x^i(C),x^j(C)\right\} \equiv \arg\max_{\{c^i,c^j\}} \varphi u(c^i)+(1-\varphi)v(c^j) \ \text{ s.t. } \ c^i+c^j=C$$

Note that by Pareto optimality, $x^i(C)$ and $x^j(C)$ are both weakly increasing functions of $C$.

Given the functions $x^1(C)$ and $x^2(C)$ we may write the value function of the planner $U(C)$ as follows

$$U\left(C\right)=\varphi u(x^i(C))+\left(1-\varphi\right)v(x^j(C)) \tag{2}$$

The question is whether the function $U\left(C\right)$ is a valid utility function: strictly increasing and strictly concave.

First, $U\left(\cdot\right)$ is clearly an increasing function of $C$, since $u$ and $v$ are strictly increasing functions and $x^i$ and $x^j$ are also increasing functions.

We then need to check if this function is concave. Consider two aggregate consumption levels $C, C'$. To show that $U$ is strictly concave, we must prove that

$$\gamma U(C)+(1-\gamma)U(C') < U(\gamma C+(1-\gamma)C')$$

for any $\gamma\in(0,1)$.

First, from (2) we may write

$$\begin{aligned}\gamma U(C)+(1-\gamma)U(C') &= \gamma\left[\varphi u(x^i(C))+(1-\varphi)v(x^j(C))\right]\\ &\quad +(1-\gamma)\left[\varphi u(x^i(C'))+(1-\varphi)v(x^j(C'))\right]\end{aligned}$$

Rearranging, we get

$$\begin{aligned}\gamma U(C)+(1-\gamma)U(C') &= \varphi\left[\gamma u(x^i(C))+(1-\gamma)u(x^i(C'))\right] \\ &\quad +(1-\varphi)\left[\gamma v(x^j(C))+(1-\gamma)v(x^j(C'))\right]\end{aligned}$$

Strict concavity of $u$ and $v$ implies

$$\begin{aligned}&\varphi\left[\gamma u(x^i(C))+(1-\gamma)u(x^i(C'))\right]+(1-\varphi)\left[\gamma v(x^j(C))+(1-\gamma)v(x^j(C'))\right] \\ &< \varphi u(\gamma x^i(C)+(1-\gamma)x^i(C'))+(1-\varphi)v(\gamma x^j(C)+(1-\gamma)x^j(C'))\end{aligned}$$

Finally, by definition of the functions $x^i$ and $x^j$, we have that

$$\begin{aligned}&\varphi u(\gamma x^i(C)+(1-\gamma)x^i(C'))+(1-\varphi)v(\gamma x^j(C)+(1-\gamma)x^j(C')) \\ \leq\ &\varphi u(x^i(\gamma C+(1-\gamma)C'))+(1-\varphi)v(x^j(\gamma C+(1-\gamma)C'))\end{aligned}$$

This last inequality follows from the fact that the consumption allocation given by

$$c^i=\gamma x^i(C)+(1-\gamma)x^i(C') \qquad \text{and} \qquad c^j=\gamma x^j(C)+(1-\gamma)x^j(C')$$

is a feasible allocation when aggregate consumption is equal to $\gamma C+(1-\gamma)C'$. However, this feasible allocation must yield weakly less welfare than the Pareto optimal allocation, by definition of Pareto optimality. Therefore, we have proven that

$$\gamma U(C)+(1-\gamma)U(C')<U(\gamma C+(1-\gamma)C'),$$

so that $U$ is a strictly concave function of $C$.

As a result, the economy admits a representative agent with utility given by $U(C)$. Thus, as long as markets are complete, even if there is heterogeneity in individual preferences the economy behaves in the aggregate *as if* there exists a representative household. This explains why when we write down models with a representative household making aggregate investment, savings, and consumption decisions, we are implicitly assuming that there are complete markets underlying the framework: while there could be many households with different utility functions making individual choices, complete markets gives rise to the existence of a representative household making these aggregate decisions.

Finally note that by the envelope condition we have that the representative's household's marginal value of consumption is equal to each individual household's marginal utility of consumption (times their pareto weight).

$$U'(C)=\varphi u'(c^i)=(1-\varphi)v'(c^j)$$

# 3 Application to Asset Pricing

Much of what we know about asset pricing is within the complete markets paradigm. This is because asset pricing becomes relatively easy when we break assets into sequences of history-contingent claims. That is, we can look at how an asset is composed of AD securities, and then just price this asset using the AD prices.

*Pricing redundant assets.* Let $\left\{d(s^t)\right\}_{t=0}^{\infty}$ be a stream of claims on time $t$, history $s^t$ consumption, where $d(s^t)$ is a function measurable in $s^t$. The price at time 0 of an asset entitling the owner to this stream of claims must be given by

$$p_0 = \sum_{t=0}^{\infty} \sum_{s^t} q(s^t) d(s^t)$$

If this equation did not hold, someone could make unbounded profits by synthesizing this asset through purchases or sales of history-contingent dated commodities and then either buying or selling the asset.

*Riskless Consol.* For example, consider the price of a riskless consol: an asset offering to pay one unit of consumption for sure in every period. Then $d\left(s^t\right) = 1$ for all $t$ and $s^t$, so the price of this asset at time 0 is given by

$$p_0 = \sum_{t=0}^{\infty} \sum_{s^t} q(s^t)$$

*Tail Assets.* We next consider tail assets. Consider the stream of dividends $\left\{d(s^t)\right\}_{t=0}^{\infty}$ and consider an arbitrary time period $\tau \geq 1$. Suppose we strip off the first $\tau - 1$ periods of the dividend. We now want to get the time 0 value of the dividend stream following this date.

In principle we would like this asset value for each possible realization of $s^\tau$. This is called a tail asset–it is the tail of the dividend process after a particular history that has been realized up to time $\tau$.

$$\{d(s^t)\}_{s^t|s^\tau}$$

The time 0 price of the asset that entitles the owner to the dividend stream $\{d(s^t)\}_{s^t|s^\tau}$ after time $\tau$, if history $s^\tau$ is realized, is given by

$$p_0 = \sum_{t \geq \tau} \sum_{s^t|s^\tau} q(s^t) d(s^t)$$

where the summation over $s^t|s^\tau$ means that we sum over all possible histories in which the first $\tau$ periods the history $s^\tau$ was realized. The units of the price are time 0 goods per unit (the numeraire) because remember we assumed that $q(s_0) = 1$.

To convert the price into units of time $\tau$, history $s^\tau$ consumption goods, you just need to divide by the price of those particular goods, that is, $q(s^\tau)$:

$$p(s^\tau) \equiv \frac{p_0}{q(s^\tau)} = \sum_{t\geq\tau}\sum_{s^t|s^\tau} \frac{q(s^t)}{q(s^\tau)} d(s^t)$$

Let

$$q_\tau(s^t) \equiv \frac{q(s^t)}{q(s^\tau)}$$

Using the FOCs of the households (1), we have that this must satisfy

$$\begin{aligned} q_\tau\left(s^t\right) &= \frac{\beta^t \pi\left(s^t\right) u'\left(c^i\left(s^t\right)\right)}{\beta^\tau \pi\left(s^\tau\right) u'\left(c^i\left(s^\tau\right)\right)} \\ &= \beta^{t-\tau}\pi\left(s^t|s^\tau\right) \frac{u'\left(c^i\left(s^t\right)\right)}{u'\left(c^i\left(s^\tau\right)\right)} \end{aligned} \tag{3}$$

This is the price of one unit of consumption delivered at time $t$ history $s^t$ in terms of the date $\tau$ history $s^\tau$ consumption good. Thus, the price at time $\tau$ for this tail asset may be written as

$$p\left(s^\tau\right) = \sum_{t\geq\tau}\sum_{s^t|s^\tau} q_\tau(s^t) d(s^t)$$

This is useful because it allows us to understand equity prices as the price of tail assets. An equity purchased at time $\tau$ entitles the owner to the dividends from time $\tau$ onward.

*Pricing one-period ahead returns.* The one-period version of equation (3) is given by

$$q_t(s^{t+1}) = \beta\pi\left(s^{t+1}|s^t\right) \frac{u'(c^i(s^{t+1}))}{u'(c^i(s^t))}$$

These are called one-period ahead "state prices."

Consider a random payoff $x(s_{t+1})$ next period. If we want to find the price at time $t$, history $s^t$ of a claim to this random payoff, we would write

$$\begin{aligned} p(s^t) &= \sum_{s^{t+1}|s^t} q_t(s^{t+1}) x(s_{t+1}) \\ &= \sum_{s^{t+1}|s^t} \beta\pi(s^{t+1}|s^t)\frac{u'(c^i(s^{t+1}))}{u'(c^i(s^t))} x(s_{t+1}) \\ &= \mathbb{E}\left[\beta\frac{u'(c(s^{t+1}))}{u'(c(s^t))} x(s_{t+1}) \middle| s^t\right] \end{aligned}$$

where I have dropped the $i$ superscript on consumption with the understanding that this equation is true for any household $i$.

Let us define

$$R(s^{t+1}) \equiv \frac{x(s_{t+1})}{p(s^t)} \tag{4}$$

as the one period gross return on the asset. Then for any asset we have that

$$1 = \mathbb{E}\left[\beta \frac{u'(c(s^{t+1}))}{u'(c(s^t))} R(s^{t+1}) \middle| s^t\right]$$

or equivalently

$$u'(c(s^t)) = \mathbb{E}\left[\beta R(s^{t+1}) u'(c(s^{t+1})) \middle| s^t\right].$$

But this is our standard Euler equation! This just tells us that the marginal utility today from an extra unit of consumption is equal to the expected marginal utility tomorrow discounted by the discount factor $\beta$, times the stochastic rate of return on the asset. Therefore, we have re-derived the Euler equation. The slight difference here is that now we see that the Euler equation must actually hold for *any* asset with stochastic return $R\left(s^{t+1}\right)$.

*The Stochastic Discount Factor.* Next, let us define a new object $m(s^{t+1})$ as the ratio of the state price to its conditional probability

$$m(s^{t+1}) \equiv \frac{q_t\left(s^{t+1}\right)}{\pi\left(s^{t+1}|s^t\right)}$$

We thus have that

$$m(s^{t+1}) = \beta \frac{u'(c(s^{t+1}))}{u'(c(s^t))} \tag{5}$$

We call $m$ the "stochastic discount factor."[1]

We can rewrite our pricing equations as follows

$$p(s^t) = \mathbb{E}\left[m(s^{t+1}) x(s_{t+1}) \middle| s^t\right] \tag{6}$$

or, equivalently

$$1 = \mathbb{E}\left[m(s^{t+1}) R(s^{t+1}) \middle| s^t\right] \tag{7}$$

This latter equation is a restriction on the conditional moments of returns and the stochastic discount factor $m_{t+1}$.

One can think of the Consumption CAPM as a particular model for the stochastic discount factor. It assumes complete markets, which implies that we have a representative consumer and hence we can use aggregate consumption in the above equations. (It also assumes Von Neumann Morgenstern time-separable utility.)

The "stochastic discount factor" $m$ shows us how the price discounts the asset's payoffs $x(s_{t+1})$ in each state. Note that if the agent were risk neutral, then $m = \beta$. Thus, the agent would discount the asset payoffs simply by the discount factor $\beta$. However, since the agent's utility has some curvature, how the agent values payoffs in each state depends on the agent's marginal utility of consumption in that state (or more precisely, the ratio of marginal utility of consumption in that state to marginal utility of consumption today). Thus, $m$ is "stochastic" in that its value depends on the realized state.

[1] In finance they may also call this object the "state price density" or the "pricing kernel."

Note that equations (6) and (7) must hold for any asset. In this context, these equations say something deep: one can incorporate all risk corrections by defining a single stochastic discount factor, the same one for each asset, and inserting it inside the expectation.

# 4 Implications of the Consumption CAPM

So far we have that for any asset with payoffs $x(s_{t+1})$, it must obey the following pricing equation in (6) and equivalently, (7).

Much of finance is based off of these equations. A few simple rearrangements and manipulations of the equations (6) and (7) provides a lot of information and introduces some classic issues in finance.

To conserve on notation, let me write $m_{t+1} \equiv m(s^{t+1})$ with the understanding that this is a function measurable in $s^{t+1}$; allow me to do the same for consumption, $c_{t+1} = c(s^{t+1})$. That is, these are stochastic variables.

## 4.1 The Risk-Free Rate.

Let's first consider the risk-free rate between today and tomorrow, which I will denote by $R^f_{t+1} = 1 + r^f_{t+1}$. We call $R^f_{t+1}$ the gross rate of return, and $r^f_{t+1}$ the net rate of return. The risk free rate is given by

$$1 = \mathbb{E}[m_{t+1} R^f_{t+1} | s^t] = \mathbb{E}[m_{t+1}|s^t] R^f_{t+1}.$$

Thus, the risk free rate must satisfy

$$R^f_{t+1} = \frac{1}{\mathbb{E}[m_{t+1}|s^t]} \tag{8}$$

**No Uncertainty.** Consider the case in which there is no uncertainty and utility is homothetic

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

In this case the stochastic discount factor satisfies

$$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma}$$

Plugging this into our equation for the interest rate and considering the case with zero uncertainty we get:

$$R^f_{t+1} = \frac{1}{\beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma}} = \frac{1}{\beta} \left( \frac{c_{t+1}}{c_t} \right)^{\gamma}$$

which we may rewrite as

$$\frac{c_{t+1}}{c_t} = \left( \beta R^f_{t+1} \right)^{1/\gamma}$$

This is the Euler equation with no uncertainty. The same intuition about intertemporal substitution holds. First, consumption growth depends on the real interest rate. If real interest rates are high relative to the agent's discount factor, then this induces the agent to save and hence experience consumption growth. Consumption growth is positive when $\beta R^f_{t+1} > 1$ and negative when the opposite is true.

Second, consumption growth is more sensitive to movements in the risk-free rate when the elasticity of intertemporal substitution, $1/\gamma$, is large. If the utility function is highly curved (high $\gamma$), the representative agent cares more about maintaining a smooth consumption profile and is less willing to rearrange consumption over time in response to interest rate incentives. It thus takes a higher interest rate change to induce a given consumption growth path. On the other hand, if the representative agent's utility is nearly linear (low $\gamma$), then the agent is more willing to rearrange consumption over time in response to interest rate movements. With very elastic intertemporal substitution, consumption growth is more sensitive to interest rate changes.

## 4.2 Risk Corrections

Using the definition of covariance

$$Cov(m, x) = \mathbb{E}[mx] - \mathbb{E}[m]\mathbb{E}[x],$$

we can rewrite (6) as

$$p_t = \mathbb{E}[m_{t+1}|s^t]\mathbb{E}[x_{t+1}|s^t] + Cov[m_{t+1}, x_{t+1}|s^t]$$

Using the fact that the risk-free rate satisfies (8), we obtain that the price at time $t$ satisfies:

$$p_t = \frac{\mathbb{E}[x_{t+1}|s^t]}{R^f_{t+1}} + Cov[m_{t+1}, x_{t+1}|s^t]$$

The first term is the standard discounted present-value formula. This is the asset's price in a risk-neutral world–if utility were linear (if the investor were risk neutral). The second term is a risk adjustment. An asset whose payoff covaries *positively* with the stochastic discount factor has a higher price. An asset whose payoff covaries negatively with the stochastic discount factor has a lower price.

To understand the risk adjustment, recall that the CCAPM has that the stochastic discount factor is given by (5). Substituting this in, we get

$$\begin{aligned} p_t &= \frac{\mathbb{E}[x_{t+1}|s^t]}{R^f_{t+1}} + Cov\left[\beta\frac{u'(c_{t+1})}{u'(c_t)}, x_{t+1}\middle| s^t\right] \\ p_t &= \frac{\mathbb{E}[x_{t+1}|s^t]}{R^f_{t+1}} + \frac{\beta}{u'(c_t)}Cov[u'(c_{t+1}), x_{t+1}|s^t] \end{aligned}$$

Remember that marginal utility $u'(c_{t+1})$ is decreasing in $c_{t+1}$. Thus

$$sign\left(Cov[u'(c_{t+1}), x_{t+1}|s^t]\right) = -sign\left(Cov[c_{t+1}, x_{t+1}|s^t]\right)$$

An asset's price is lower if its payoffs covary positively with consumption. Conversely, an asset's price is higher if it its payoffs covary negatively with consumption. Therefore, payoffs that are positively correlated with consumption growth have lower prices, to compensate investors for risk.

The intuition for this is as follows. Investors do not like uncertainty about consumption. If you buy an asset whose payoff covaries positively with consumption, i.e it pays off well when you are already feeling wealthy, and pays off badly when you are already feeling poor, that asset will make your consumption stream more volatile. You don't like this, thus you would require a low price to induce you to buy such an asset.

Conversely, if you buy an asset whose payoff covaries negatively with consumption, it helps to smooth consumption. You like this asset–it is more valuable than just its expected payoff might indicate, and hence you would be willing to pay a higher price for you to hold this asset. Insurance is an example of this. Insurance pays off exactly when your wealth and consumption would otherwise be low–you get a check when your house burns down. For this reason, you are happy to hold insurance even though you pay an insurance premium on it. That is, you are happy to buy insurance even though the price of insurance is greater than its expected value.

**Risk Corrections in terms of returns.** It is worth restating the same intuition in terms of returns. Start with the basic pricing equation for the return on some asset $j$

$$1 = \mathbb{E}[m_{t+1} R^j(s^{t+1}) | s^t]$$

where $R^j(s^{t+1})$ denotes the realized return on asset $j$ in history $s^t$. Again for notation's sake, let me write $R^j_{t+1} \equiv R^j(s^{t+1})$ with the understanding that this is a function measurable in $s^{t+1}$. That is, the returns on this asset are stochastic.

The asset pricing model says that, although expected returns can vary across time and assets, expected *discounted* returns should always be the same and equal to 1. Applying the covariance decomposition, we get

$$1 = \mathbb{E}[m_{t+1}|s^t]\mathbb{E}[R^j_{t+1}|s^t] + Cov[m_{t+1}, R^j_{t+1}|s^t]$$

Dividing both sides by $\mathbb{E}[m_{t+1}|s^t]$,

$$\frac{1}{\mathbb{E}[m_{t+1}|s^t]} = \mathbb{E}[R^j_{t+1}|s^t] + \frac{1}{\mathbb{E}[m_{t+1}|s^t]} Cov[m_{t+1}, R^j_{t+1}|s^t]$$

Again using that the risk-free rate satisfies (8), we get

$$R^f_{t+1} = \mathbb{E}[R^j_{t+1}|s^t] + \frac{Cov[m_{t+1}, R^j_{t+1}|s^t]}{\mathbb{E}[m_{t+1}|s^t]}$$

Thus

$$\mathbb{E}[R^j_{t+1}|s^t] - R^f_{t+1} = -\frac{Cov[m_{t+1}, R^j_{t+1}|s^t]}{\mathbb{E}[m_{t+1}|s^t]}$$

Next, substituting in for $m$, we have that

$$\mathbb{E}[R^j_{t+1}|s^t] - R^f_{t+1} = -\frac{Cov[u'(c_{t+1}), R^j_{t+1}|s^t]}{\mathbb{E}[u'(c_{t+1})|s^t]} \tag{9}$$

The left hand side is called "excess returns" and is equal to the expected return minus the risk free rate. All assets have an expected return equal to the risk free rate plus a risk adjustment. Assets whose returns covary positively with consumption make consumption more volatile, and so must promise higher expected returns in order to induce investors to hold them. Conversely assets that covary negatively with consumption, such as insurance, can offer expected rates of return that are lower than the risk free rate (or even negative net expected returns).

Much of finance focuses on expected returns. We think of expected returns increasing or decreasing to clear markets for securities. The intuition is that "riskier" securities must offer higher expected returns in order to induce investors to hold them. This is the same as saying that "riskier" securities must trade at lower initial prices in order to induce investors to hold them. On the other hand, less risky securities can offer lower returns.

Bottom line: it is not the variance per se that matters, it is the *covariance* that matters!

**Zero Covariance assets.** When we say "risky" securities, we mean securities that *covary* with the marginal utility of consumption. That is, you might think that an asset with a volatile (high variance) payoff is risky and thus should have a large risk correction. However, this is not necessarily true; what matters is the *covariance* with consumption growth, not the actual variance.

Suppose an asset has payoffs with high variance $\sigma^2(x)$, but this payoff is uncorrelated with the discount factor $m$. In this case the asset receives no risk correction in its price, and pays an expected return equal to the risk-free rate! That is, if for any asset $j$ such that

$$Cov[m_{t+1}, R^j_{t+1}|s^t] = 0,$$

the expected return is simply the risk-free rate

$$\mathbb{E}[R^j_{t+1}|s^t] = R^f_{t+1}$$

and there are zero excess returns. Similarly, the price at time $t$ is equal to the present discounted value of expected payoffs

$$p_t = \frac{\mathbb{E}[x_{t+1}|s^t]}{R^f_{t+1}}$$

This is true no matter how large $\sigma^2(x)$. This prediction holds even if the payoff $x_{t+1}$ is highly volatile and investors are highly risk averse!

Therefore, it is only the component of a payoff that is *correlated* with the stochastic discount factor that generates an excess return. Idiosyncratic risk, uncorrelated with the stochastic discount factor, generates no premium or risk adjustment.

## 5 The Sharpe Ratio and the Hansen-Jagannathan bound

We now consider the ratio of mean excess returns to the standard deviation of returns.

$$\frac{\mathbb{E}[R^j_{t+1}|s^t] - R^f_{t+1}}{\sigma(R^j_{t+1})}$$

where $E$ is the unconditional expectations operator and $\sigma$ is the standard deviation. This ratio is called the "Sharpe Ratio." This is a more interesting characterization of a security than the mean return alone; if you borrow and put more money into a security, you can increase the mean return of your position, but you do not increase the Sharpe ratio, since the standard deviation increases at the same rate as the mean.

Interestingly there is a bound on this ratio due to Hansen and Jagannathan (1991).

**Lemma 1.** *(The Hansen-Jagannathan Bound) The Sharpe ratio of* any *portfolio cannot exceed the ratio of the standard devation of the stochastic discount factor to its mean.*

$$\frac{\sigma(m_{t+1})}{\mathbb{E}[m_{t+1}]} \geq \sup_j \left| \frac{\mathbb{E}[R^j_{t+1}|s^t] - R^f_{t+1}}{\sigma(R^j_{t+1})} \right|$$

*Proof.* Consider equation (7) and take the unconditional expectation of both sides. By the law of iterated expectations, we get the unconditional moments restriction

$$1 = \mathbb{E}[m_{t+1} R^j_{t+1}].$$

We thus write for any given asset return

$$1 = \mathbb{E}[m_{t+1}]\mathbb{E}[R^j_{t+1}] + Cov(m_{t+1}, R^j_{t+1})$$

$$1 = \mathbb{E}[m_{t+1}]\mathbb{E}[R^j_{t+1}] + \rho(m_{t+1}, R^j_{t+1})\sigma(m_{t+1})\sigma(R^j_{t+1})$$

where we let $\rho$ denote the correlation:

$$\rho(m_{t+1}, R^j_{t+1}) \equiv Corr(m_{t+1}, R^j_{t+1}) = \frac{Cov(m_{t+1}, R^j_{t+1})}{\sigma(m_{t+1})\sigma(R^j_{t+1})}$$

Dividing everything by $\mathbb{E}[m_{t+1}]$, we get

$$\frac{1}{\mathbb{E}[m_{t+1}]} = \mathbb{E}[R^j_{t+1}] + \rho(m_{t+1}, R^j_{t+1})\frac{\sigma(m_{t+1})}{\mathbb{E}[m_{t+1}]}\sigma(R^j_{t+1})$$

Hence

$$\mathbb{E}[R^j_{t+1}] - R^f_{t+1} = -\rho(m_{t+1}, R^j_{t+1})\frac{\sigma(m_{t+1})}{\mathbb{E}[m_{t+1}]}\sigma(R^j_{t+1})$$

Correlation coefficients cannot be greater than 1 in magnitude: $\rho \in [-1, 1]$. Therefore we get the following bound on mean returns

$$\left|\mathbb{E}[R^j_{t+1}] - R^f_{t+1}\right| \leq \frac{\sigma(m_{t+1})}{\mathbb{E}[m_{t+1}]}\sigma(R^j_{t+1})$$

This holds for any asset $j$. $\square$

What does the Hansen and Jagannathan (1991) bound tell us? First, given a stochastic discount factor, this inequality means that the Sharpe ratios of assets are limited by the volatility of the discount factor. That is, the set of means and variances of asset returns are limited by what is called the *mean-variance frontier*. See Figure 1. This has the following implications.

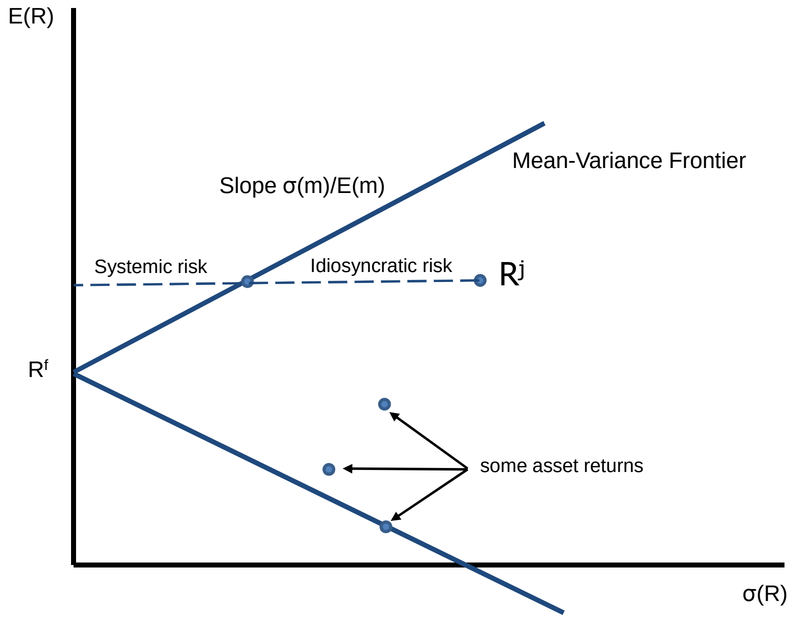


Figure 1. The Mean-Variance Frontier

1. Means and variances of asset returns must lie in the wedge-shaped region illustrated in Figure 1. The boundary of the mean-variance region in which assets can lie is called the mean-variance frontier. The slope of the mean-variance frontier is the largest available Sharpe ratio. It answers the following question: "How much more mean return can I get by taking on a bit more volatility in my portfolio?"

2. All returns on the frontier are perfectly correlated with the discount factor $m_{t+1}$. That is, the frontier is generated by

$$\left|\mathbb{E}[R^j_{t+1}] - R^f_{t+1}\right| \leq \left|\rho(m_{t+1}, R^j_{t+1})\right| \frac{\sigma(m_{t+1})}{\mathbb{E}[m_{t+1}]}\sigma(R^j_{t+1})$$

   with $|\rho(m, R)| = 1$. Returns on the upper part of the frontier are perfectly negatively correlated with $m$, and hence positively correlated with consumption. They are the maximally risky and thus yield the highest expected returns. Returns on the lower part of the frontier are perfectly positively correlated with $m$, and hence negatively correlated with consumption. These securities thus provide the best insurance against consumption and therefore yield the lowest expected returns.

3. We can plot the decomposition of a return into a priced systematic component and a residual idiosyncratic component. These components are illustrated in Figure 1. The priced "systematic" component is perfectly correlated with the discount factor. The "residual" or "idiosyncratic" component generates no expected return, so it lies flat as shown in the figure, and it is uncorrelated with the stochastic discount factor $m$.

Furthermore, this bound can be used to check the "viability" of a proposed discount factor. We consider this next.

# 6 The Equity Premium Puzzle

We now consider the Hansen and Jagannathan (1991) bound in terms of the data. For any asset:

$$\left|\frac{\mathbb{E}[R^j_{t+1}] - R^f_{t+1}}{\sigma(R^j_{t+1})}\right| \leq \frac{\sigma(m_{t+1})}{\mathbb{E}[m_{t+1}]}.$$

We can write the right hand side as

$$\frac{\sigma(m_{t+1})}{\mathbb{E}[m_{t+1}]} = \frac{\sqrt{\sigma^2(m_{t+1})}}{\mathbb{E}[m_{t+1}]} = \frac{\sqrt{\mathbb{E}[m^2_{t+1}] - (\mathbb{E}[m_{t+1}])^2}}{\mathbb{E}[m_{t+1}]} = \sqrt{\frac{\mathbb{E}[m^2_{t+1}]}{(\mathbb{E}[m_{t+1}])^2} - 1}$$

The CCAPM gives us that

$$m_{t+1} = \beta\frac{u'(c_{t+1})}{u'(c_t)}$$

With CRRA utility, we have that

$$m_{t+1} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$$

This equation becomes

$$\sqrt{\frac{\mathbb{E}[m^2_{t+1}]}{(\mathbb{E}[m_{t+1}])^2} - 1} = \sqrt{\frac{\mathbb{E}\left[\left(\frac{c_{t+1}}{c_t}\right)^{-2\gamma}\right]}{\left[\mathbb{E}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right]^2} - 1}$$

Suppose that consumption growth is log Normal.

$$\log c_{t+1} - \log c_t \sim N\left(\mu, \sigma^2\right)$$

Using the moment-generating function for the Normal distribution, we have that

$$
\begin{aligned}
\mathbb{E}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} &= \mathbb{E}\exp\left\{\log\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right\} = \mathbb{E}\exp\left\{-\gamma\log\left(\frac{c_{t+1}}{c_t}\right)\right\} \\
&= \exp\left\{-\gamma\mu + \frac{1}{2}\gamma^2\sigma^2\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\left(\frac{c_{t+1}}{c_t}\right)^{-2\gamma} &= \mathbb{E}\exp\left\{\log\left(\frac{c_{t+1}}{c_t}\right)^{-2\gamma}\right\} = \mathbb{E}\exp\left\{-2\gamma\log\left(\frac{c_{t+1}}{c_t}\right)\right\} \\
&= \exp\left\{-2\gamma\mu + \frac{1}{2}4\gamma^2\sigma^2\right\}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sqrt{\frac{\mathbb{E}\left[\left(\frac{c_{t+1}}{c_t}\right)^{-2\gamma}\right]}{\left[\mathbb{E}\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right]^2} - 1} &= \sqrt{\frac{\exp\left\{-2\gamma\mu + \frac{1}{2}4\gamma^2\sigma^2\right\}}{\exp\left\{-2\gamma\mu + \frac{1}{2}2\gamma^2\sigma^2\right\}} - 1} \\
&= \sqrt{\exp\left\{2\gamma^2\sigma^2 - \gamma^2\sigma^2\right\} - 1} \\
&= \sqrt{\exp\left\{\gamma^2\sigma^2\right\} - 1} \\
&\approx \gamma\sigma
\end{aligned}
$$

because $z^2 = \exp(\gamma^2\sigma^2) - 1$ implies $1 + z^2 = \exp(\gamma^2\sigma^2)$ so that $\log(1 + z^2) = \gamma^2\sigma^2$. Thus $z^2 \approx \gamma^2\sigma^2$.

As a result, we have that with these assumptions the Hansen-Jagannathan bound satisfies:

$$
\left|\frac{\mathbb{E}[R^j_{t+1}] - R^f_{t+1}}{\sigma(R^j_{t+1})}\right| \leq \gamma\sigma
$$

where $\gamma$ is the CRRA coefficient and $\sigma$ is the standard deviation of consumption growth. What does this mean? The slope of the mean-std deviation frontier is higher if the economy is riskier, i.e. if consumption is more volatile, or if households are more risk averse. Both situations make households more reluctant to take on the extra risk of holding risky assets, thereby driving up the risk premium.

Let's now bring this equation to the data.

**Some Facts from the Data.**

1. Real stock returns have high mean and high variance: in the post-war sample, the average aggregate real stock returns are approximately 9% per year, with a standard deviation of around 16%.

2. The average risk-free rate is low. Using 3-month T-bills as a proxy, we obtain an average

real rate of 1% per year. (Caveat: T-bills are nominal, so in real terms returns are not exactly risk-free).

3. Real consumption growth has very low volatility. Aggregate non-durables and services has a standard deviation of approximately 1%. The mean of the growth rate is approximately 2% per year.

Therefore, historical facts 1 and 2 give us that the average Sharpe ratio has been about $.5$ in the post-war period.

$$\frac{\mathbb{E}[R] - R^f}{\sigma(R)} = \frac{1.09 - 1.01}{.16} = \frac{.08}{.16} = .5$$

Therefore, with a consumption growth standard deviation of $\sigma = .01$ we have that

$$.5 \leq \gamma\,(.01)$$

We thus need a risk aversion coefficient of over 50 for this equation to hold!! This is crazy, since most micro estimates of risk aversion estimate $\gamma$ to be less than 10.

Therefore the model cannot simultaneously match the equity premium we see on stocks vs. bonds with the low consumption volatility we see in the data and most people's level of risk aversion.

This is what is known as the "Equity Premium Puzzle;" see Mehra and Prescott (1985). Either (i) people are way more risk averse than what is found in experiments, (ii) the stock returns of the last 50 years have just been largely good luck rather than an equilibrium compensation for risk, or (iii) something is deeply wrong with the model. In terms of the third posibility: it could be that the way we specify utility is incorrect or the fact that we use aggregate consumption data, i.e. we assume complete markets and a representative household.

As a result, the "Equity Premium Puzzle" has attracted the attention of a lot of research in macro-finance. In any case, it is a great example of the interplay between theory and data.

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