Brouwer's and Kakutani's Fixed Point Theorems

Jennifer La'O*

I will end this class with two more fixed point theorems which are useful in economics.

1 Brouwer's Fixed Point Theorem

We begin with an extremely simple intermediate value theorem.

Theorem 1. *(*Bolzano's Theorem *or* Bolzano's Intermediate Value Theorem*). Let* $[a, b] \subseteq \mathbb{R}$.

Let f *be a continuous function on* $[a, b]$*. If* $f(a) > 0$ *and* $f(b) < 0$*, or if* $f(a) < 0$ *and* $f(b) > 0$*, then there exists at least one* $x_0 \in (a, b)$ *such that*

$$f(x_0) = 0.$$

In other words, if a continuous function defined on an interval is sometimes positive and sometimes negative, it must be zero at some point.

We next define a fixed point of a function.

Definition 1. Given a set X and a function $f : X \to X$, we say an element $x \in X$ is a fixed point of f if $x = f(x)$.

Theorem 2. *(*Brouwer's Fixed Point Theorem.*) Let* $X = [a, b] \subseteq \mathbb{R}$*. If* $f : X \to X$ *is a continuous function then* f *has a fixed point.*

Proof. If either $f(a) = a$ or $f(b) = b$, then we are done. Otherwise $f(a) > a$ and $f(b) < b$. Define

$$g(x) = f(x) - x$$

Then $g(a) > 0$ and $g(b) < 0$. Moreover, g is continuous since f is continuous. By Bolzano's Theorem, there exists an $x^* \in X$ such that $g(x^*) = 0$. □

Theorem 3. *(*Brouwer's Fixed Point Theorem: General Version.*) Let* $X \subseteq \mathbb{R}^N$ *be nonempty, compact, and convex. If* $f : X \to X$ *is a continuous function then* f *has a fixed point.*

Brouwer's fixed point theorem can be used to prove existence of equilibrium in a pure exchange economy.

*Department of Economics, Columbia University, jenlao@columbia.edu.

2 Kakutani's Fixed Point Theorem

Brouwer's fixed point theorem deals with fixed points of continuous functions. Kakutani's fixed point theorem generalizes the theorem to correspondences.

We first define what we mean by a fixed point of a correspondence.

Definition 2. An element $x \in X$ is a fixed point of a correspondence $G : X \to X$ if $x \in G(x)$.

Theorem 4. *(*Kakutani's Fixed Point Theorem*) Let $X \subseteq \mathbb{R}^N$ be nonempty, compact, and convex and let $G : X \to X$ be a correspondence. If G is nonempty, convex-valued and upper hemi-continuous, then G has a fixed point.*

Typically we use Kakutani's fixed point theorem to prove the existence of Nash Equilibria in games.